Earth-Scattering of Dark Matter:
when Dark Matter particle physics and astrophysics collide

Bradley J. Kavanagh
LPTHE - Paris VI

Based on arXiv:1611.05453
with Riccardo Catena and Chris Kouvaris

MIAPP, Munich - 21st March 2017

bkavanagh@lpthe.jussieu.fr
@BradleyKavanagh

Dark Matter is heavier in Munich

Local DM density: $\rho_\chi \sim 0.3\ \mathrm{GeV/cm^3}$

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you get about 1 DM particle per glass…

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Local DM density: $\rho_\chi \sim 0.3\ \mathrm{GeV/cm^3}$

In the UK, for $m_\chi \sim 150\ \mathrm{GeV}$
you get about 1 DM particle per glass…

In Munich, you need $m_\chi \sim 300\ \mathrm{GeV}$
to get 1 DM particle per glass…

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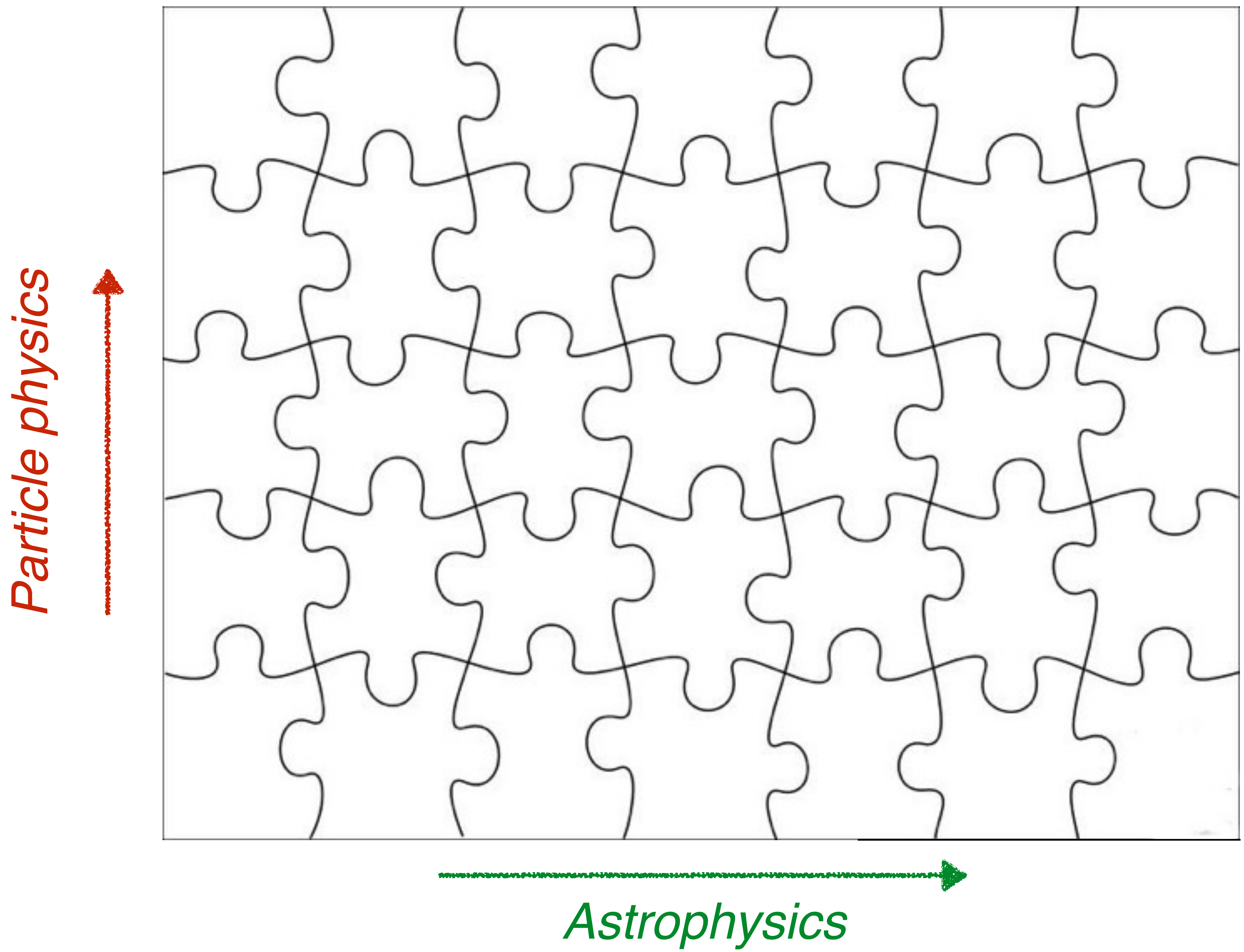
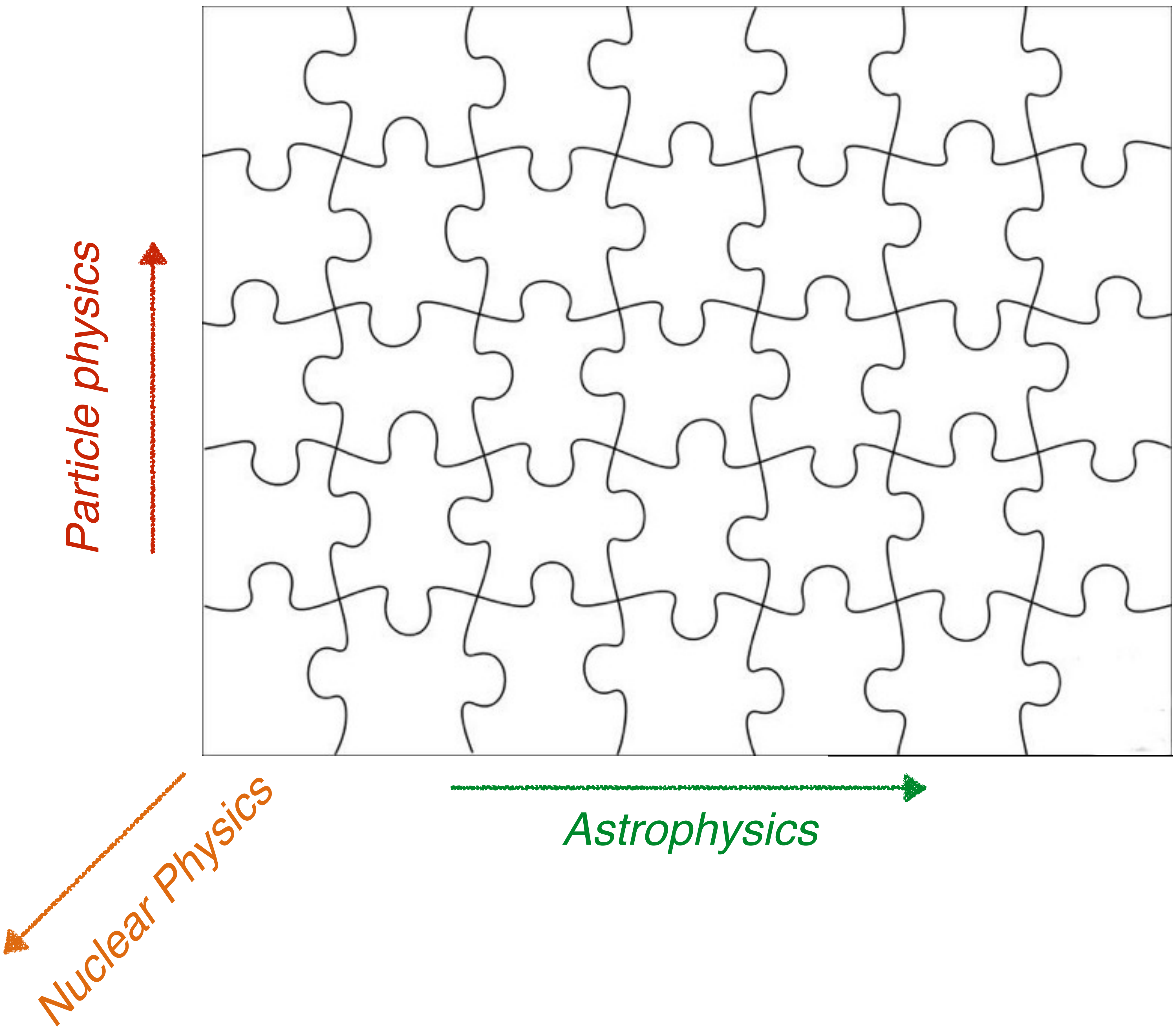
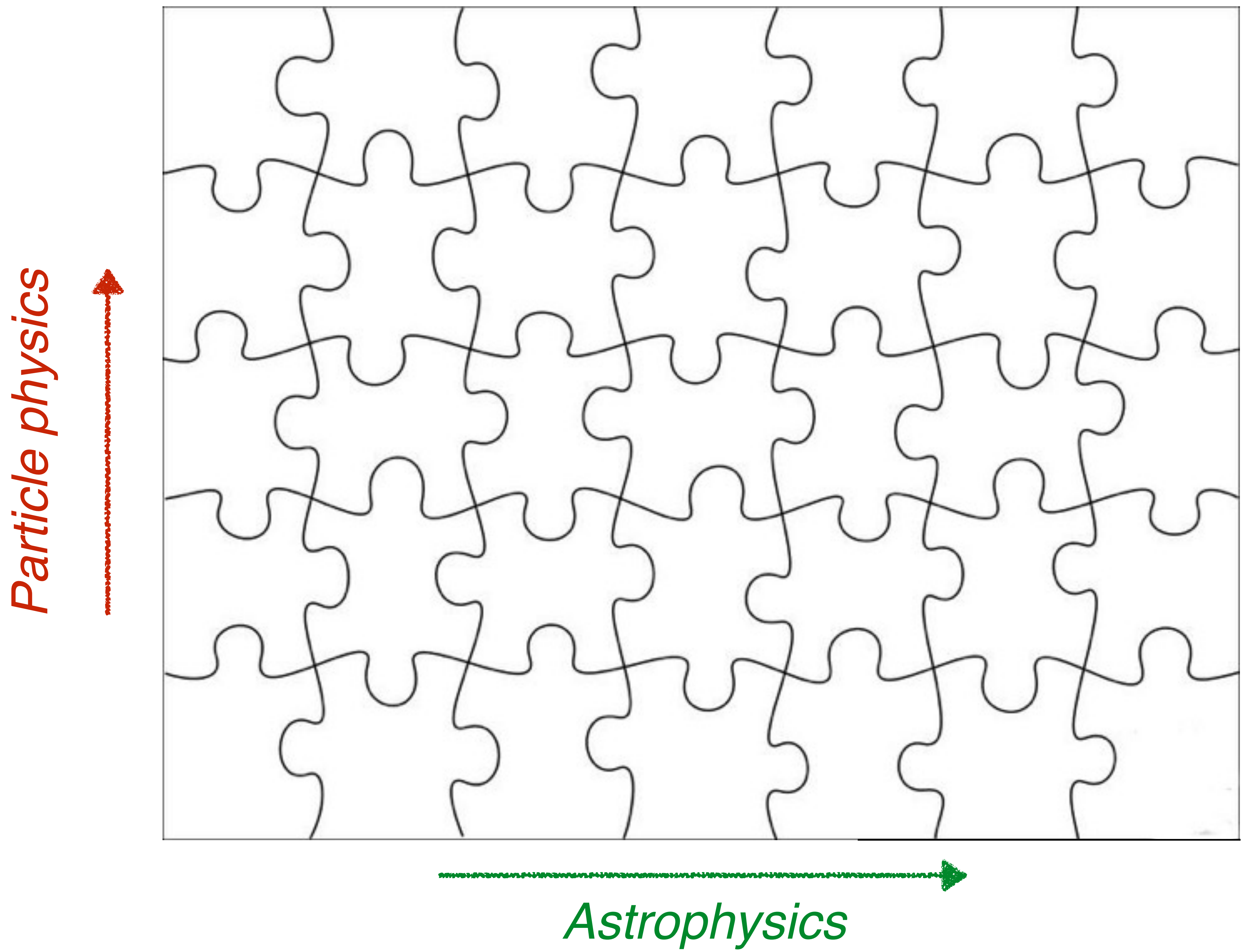

Particle physics
Astrophysics


Particle physics
Astrophysics
Nuclear Physics


Particle physics
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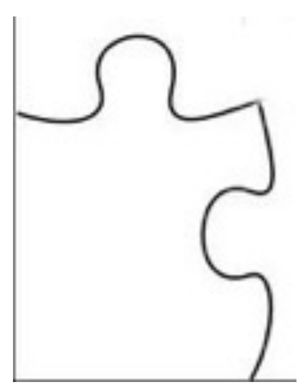
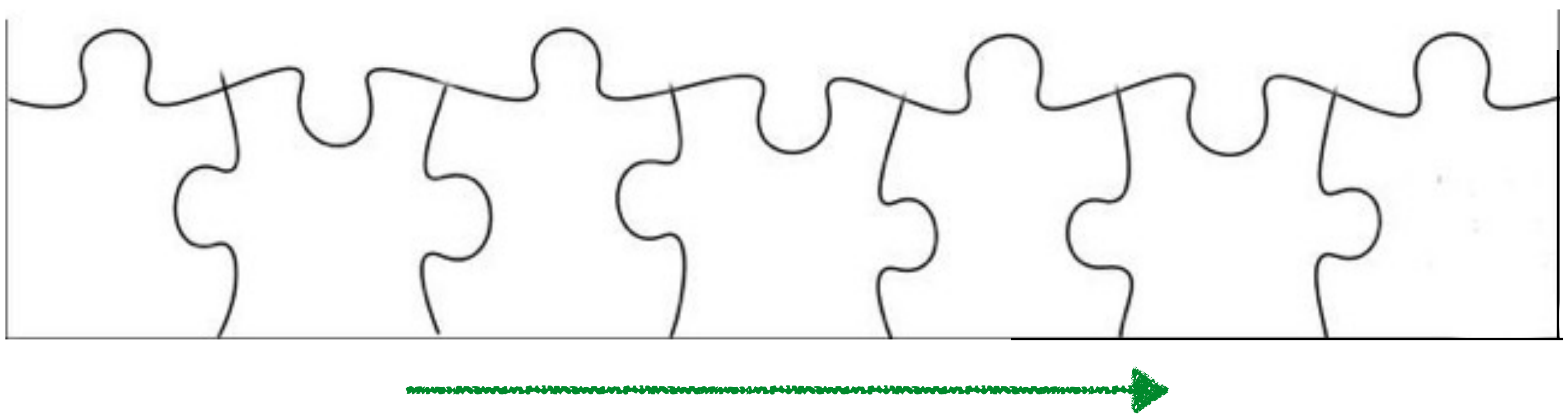


‘Standard’ SI/SD WIMPs
with SHM distribution



Astrophysics

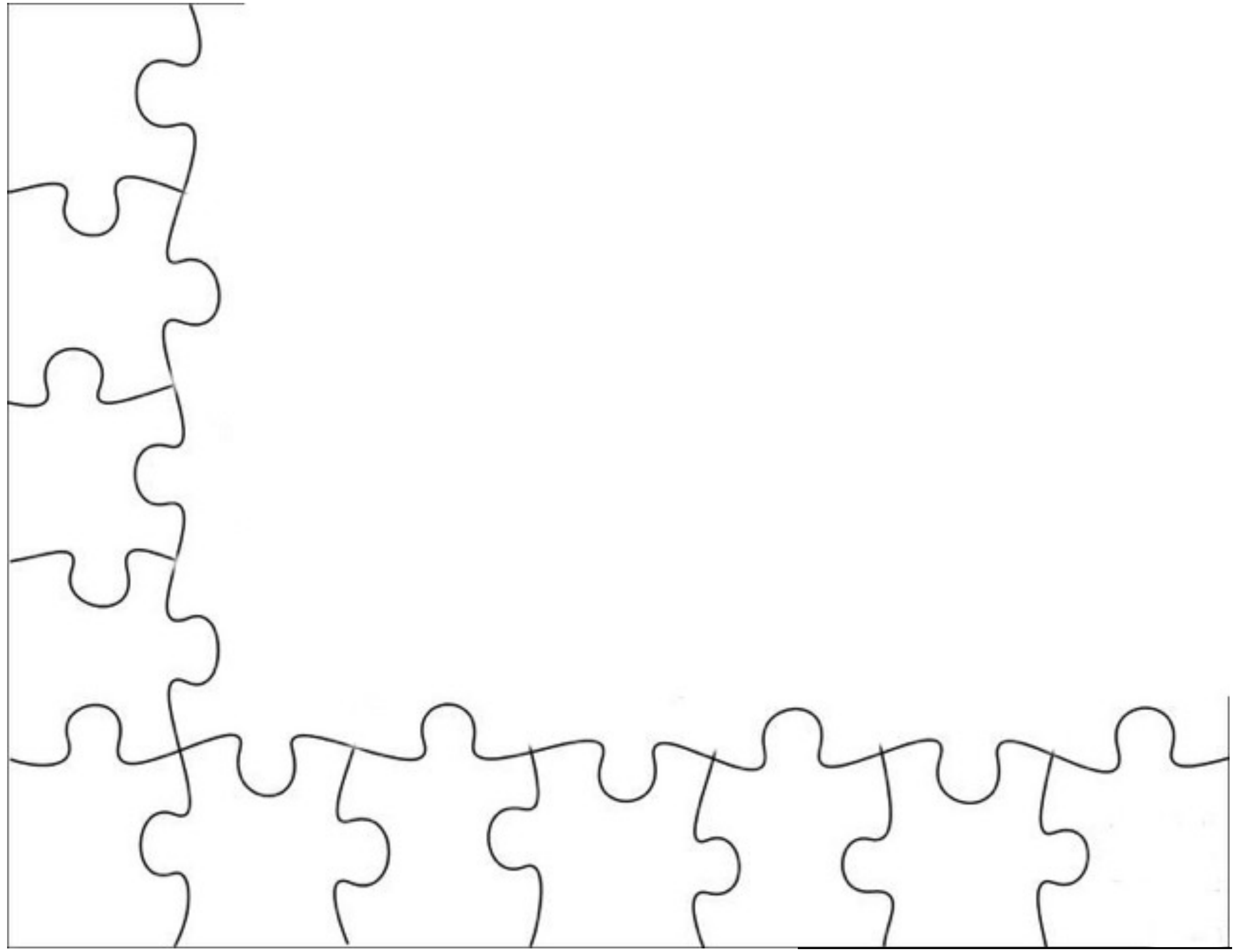
Reconstructing DM parameters without astro assumptions

1303.6868, 1312.1852, 1410.8051, 1609.08630 and others

Discriminating DM operators

1505.07406

Particle physics



DM-SM operator running and mixing

1605.04917, 1702.00016

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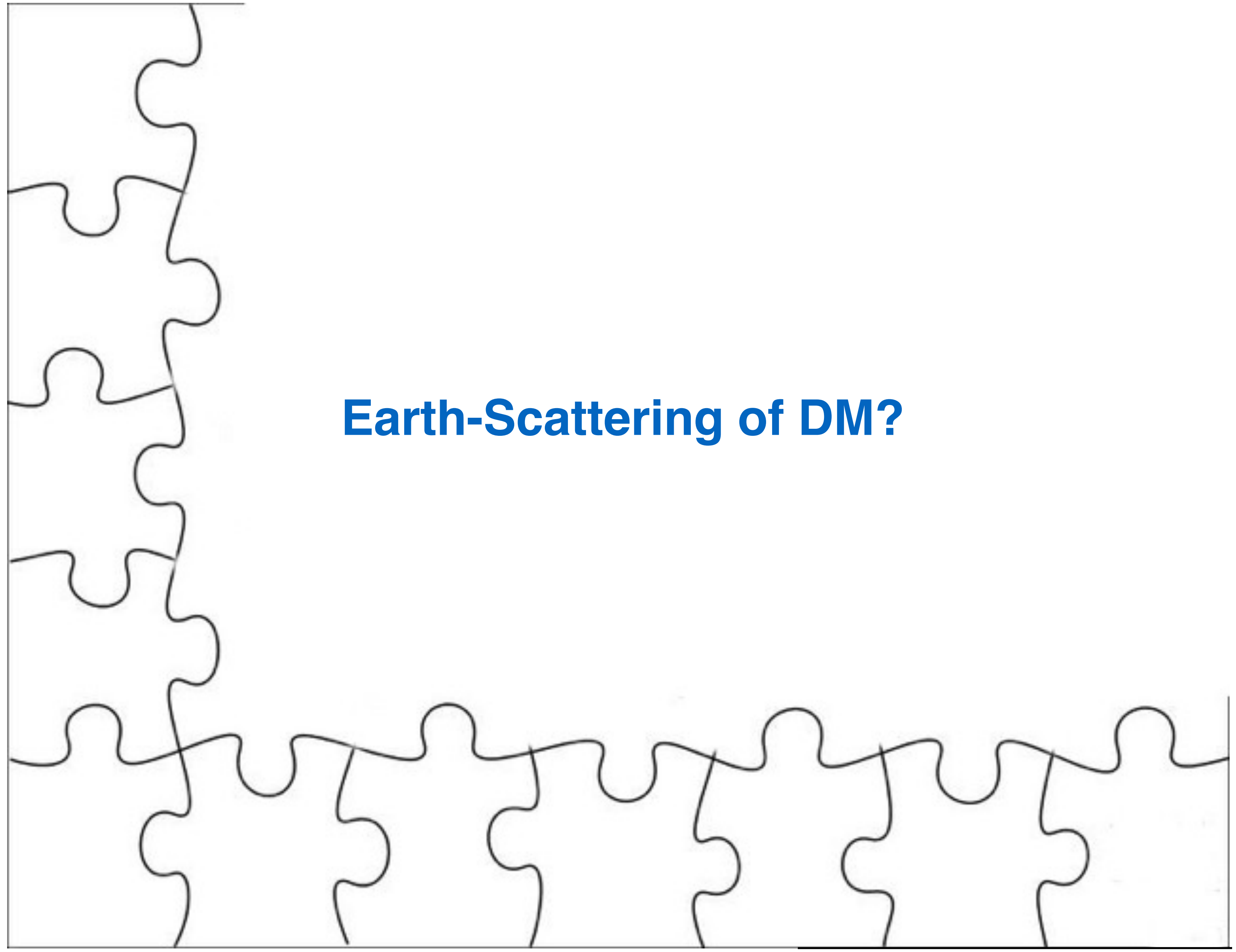

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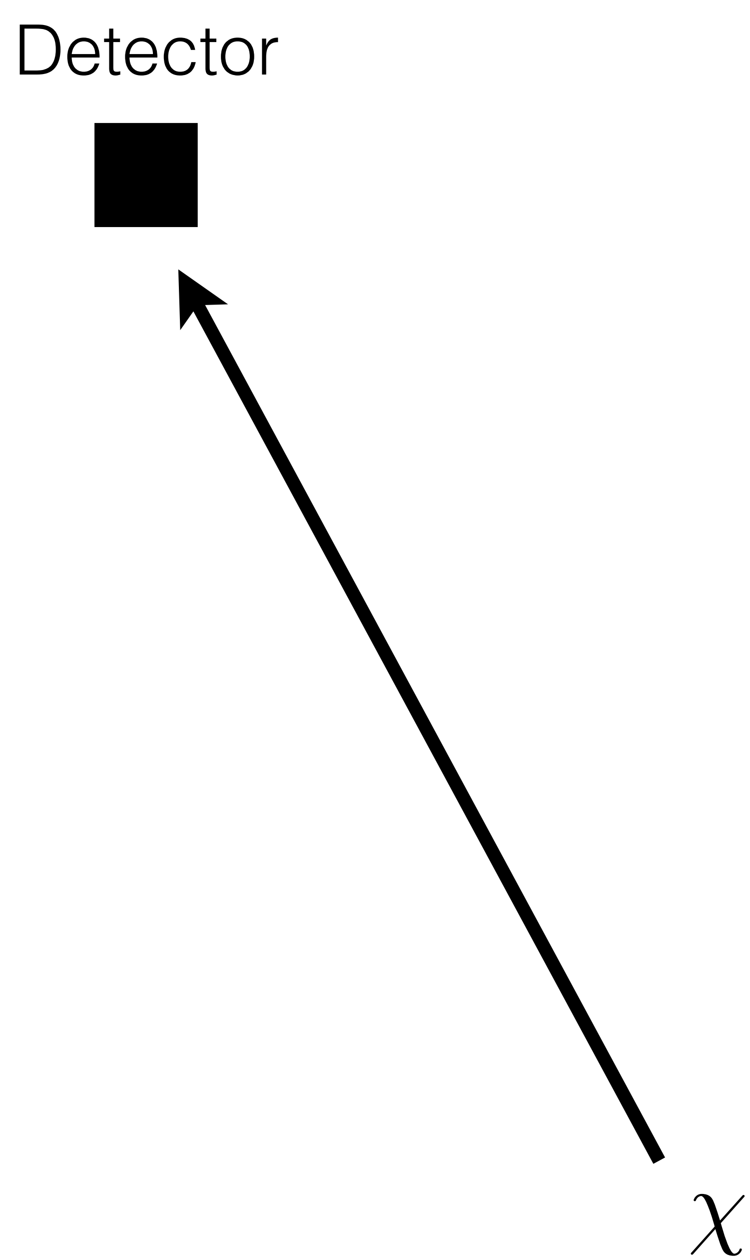
1605.04917, 1702.00016

Astrophysics

Reconstructing DM parameters without astro assumptions

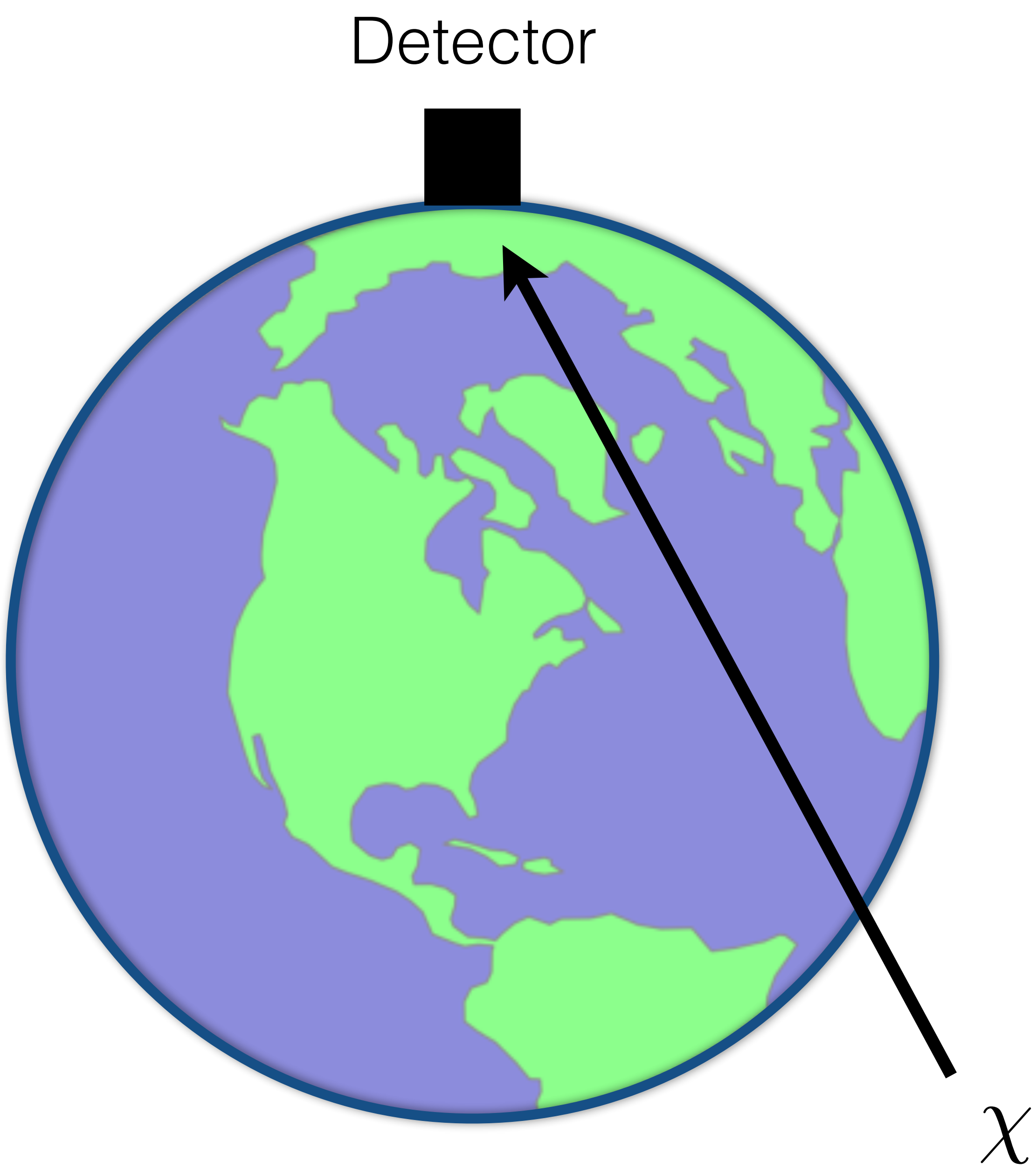
1303.6868, 1312.1852, 1410.8051, 1609.08630 and others

Direct Detection of DM (in space?)



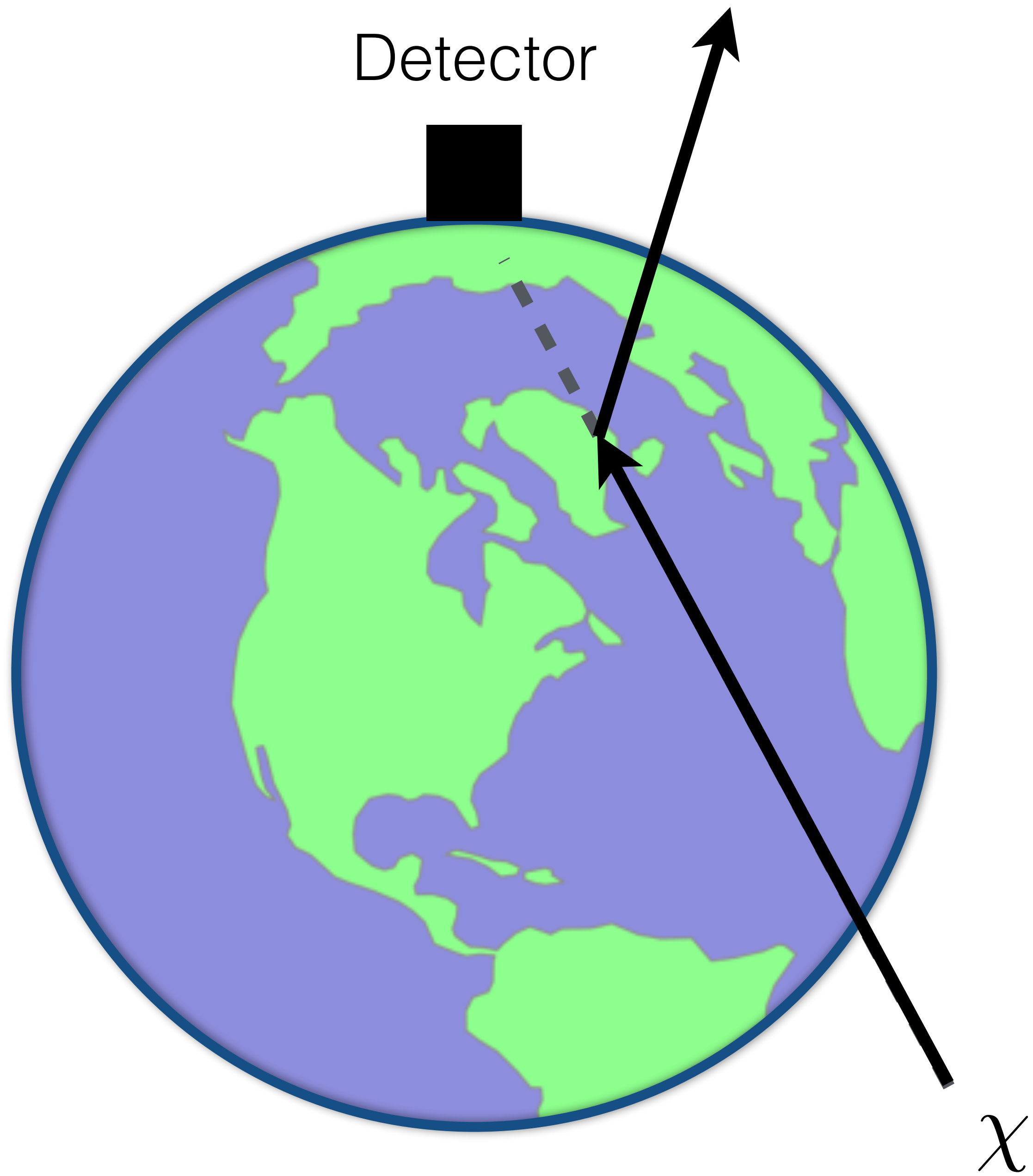
Unscattered (free) DM: $f_0(\mathbf{v})$

Direct Detection of DM on Earth



Unscattered (free) DM: $f_0(\mathbf{v})$

Earth-Scattering - Attenuation



Previous calculations usually only consider DM attenuation

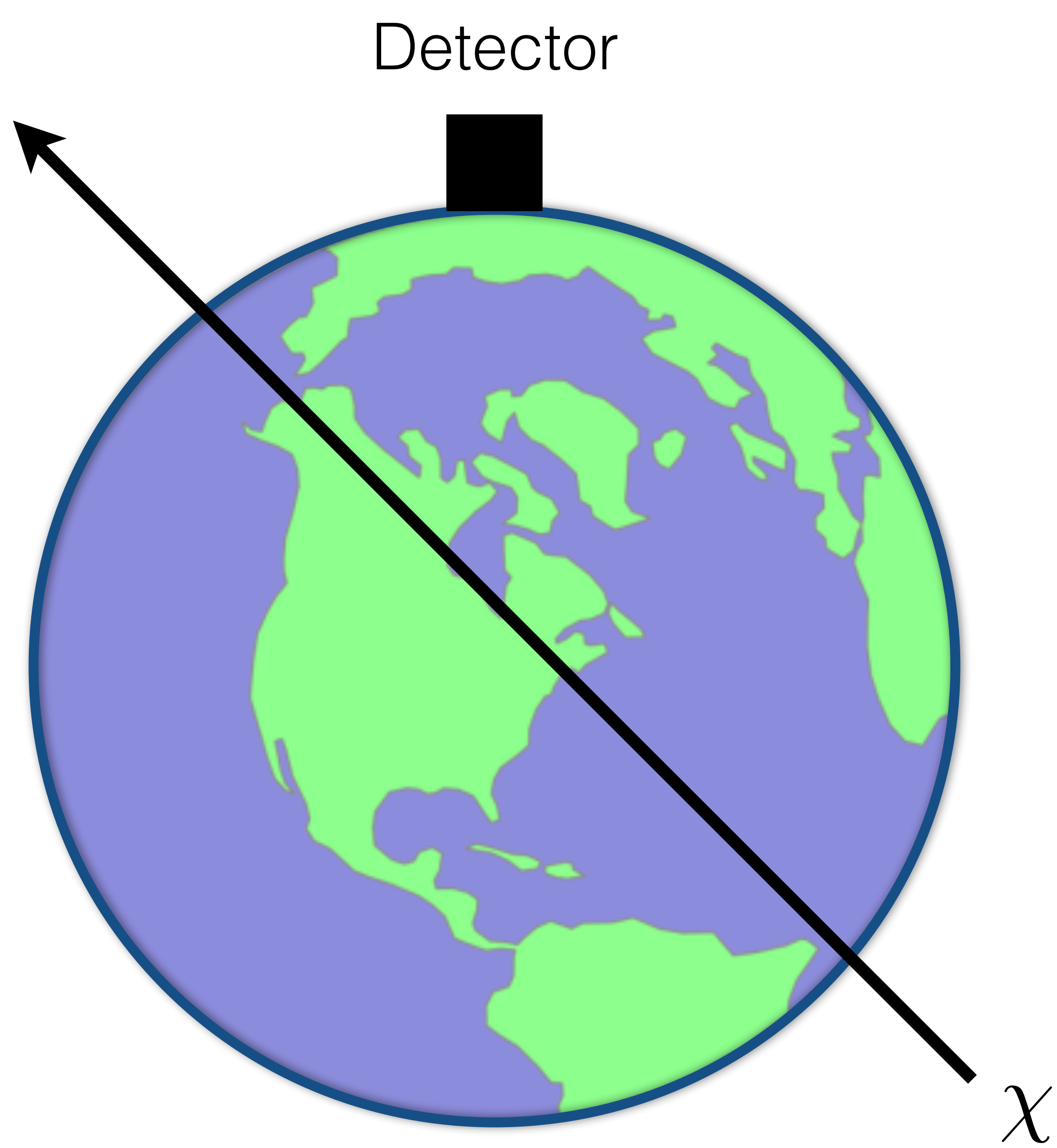
Zaharijas & Farrar [astro-ph/0406531]

Kouvaris & Shoemaker [1405.1729,1509.08720]

DAMA [1505.05336]

Attenuation of DM flux: $f(\mathbf{v}) \rightarrow f_0(\mathbf{v}) - f_A(\mathbf{v})$

Earth-Scattering - Deflection



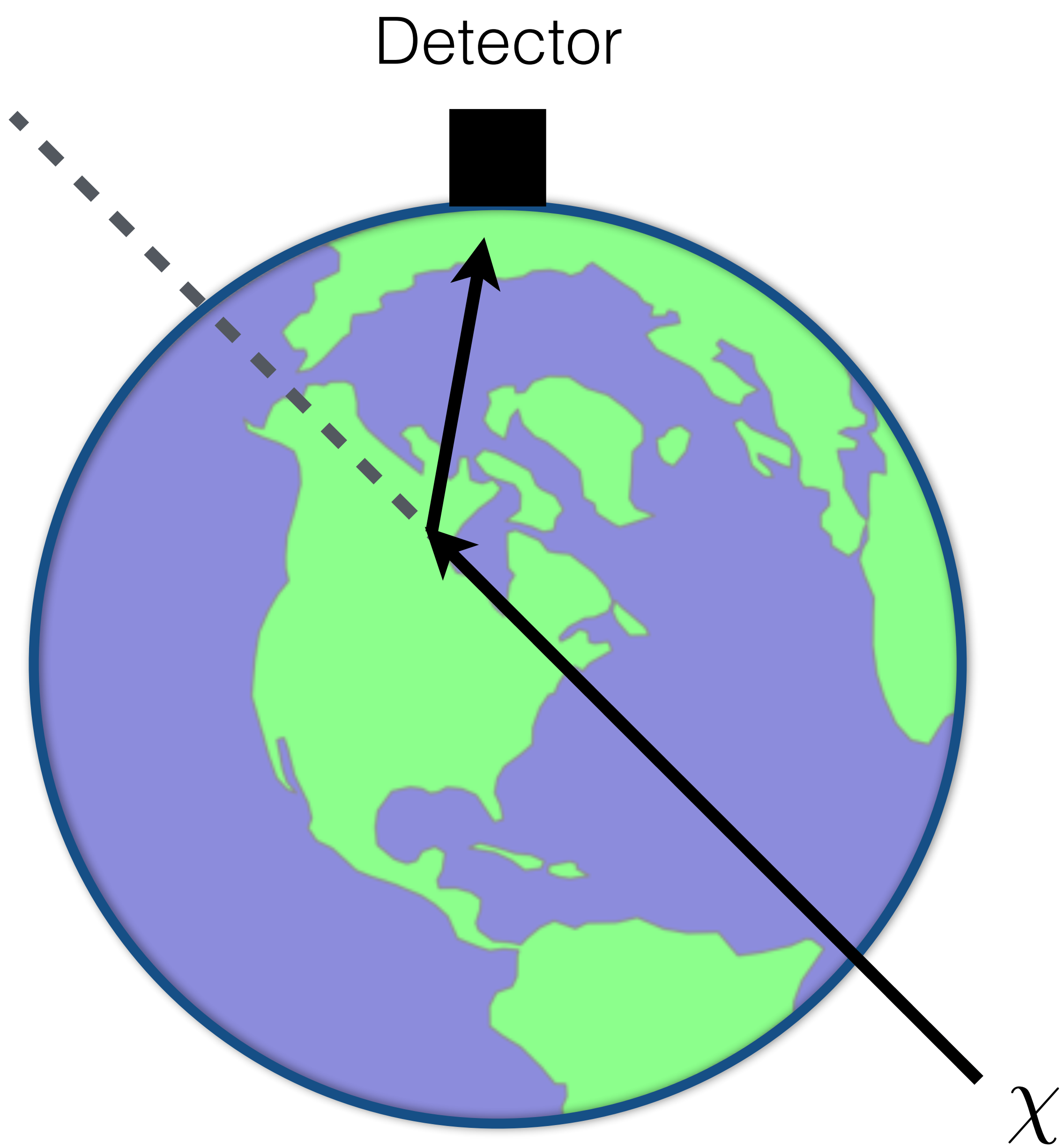
Earth-Scattering - Deflection

Considered in early Monte Carlo simulations…

Collar & Avignone [PLB 275, 1992 and others]

As well as more recent ones…

Emken, Kouvaris & Shoemaker [1702.07750] (see later)



Assuming DM mean free path

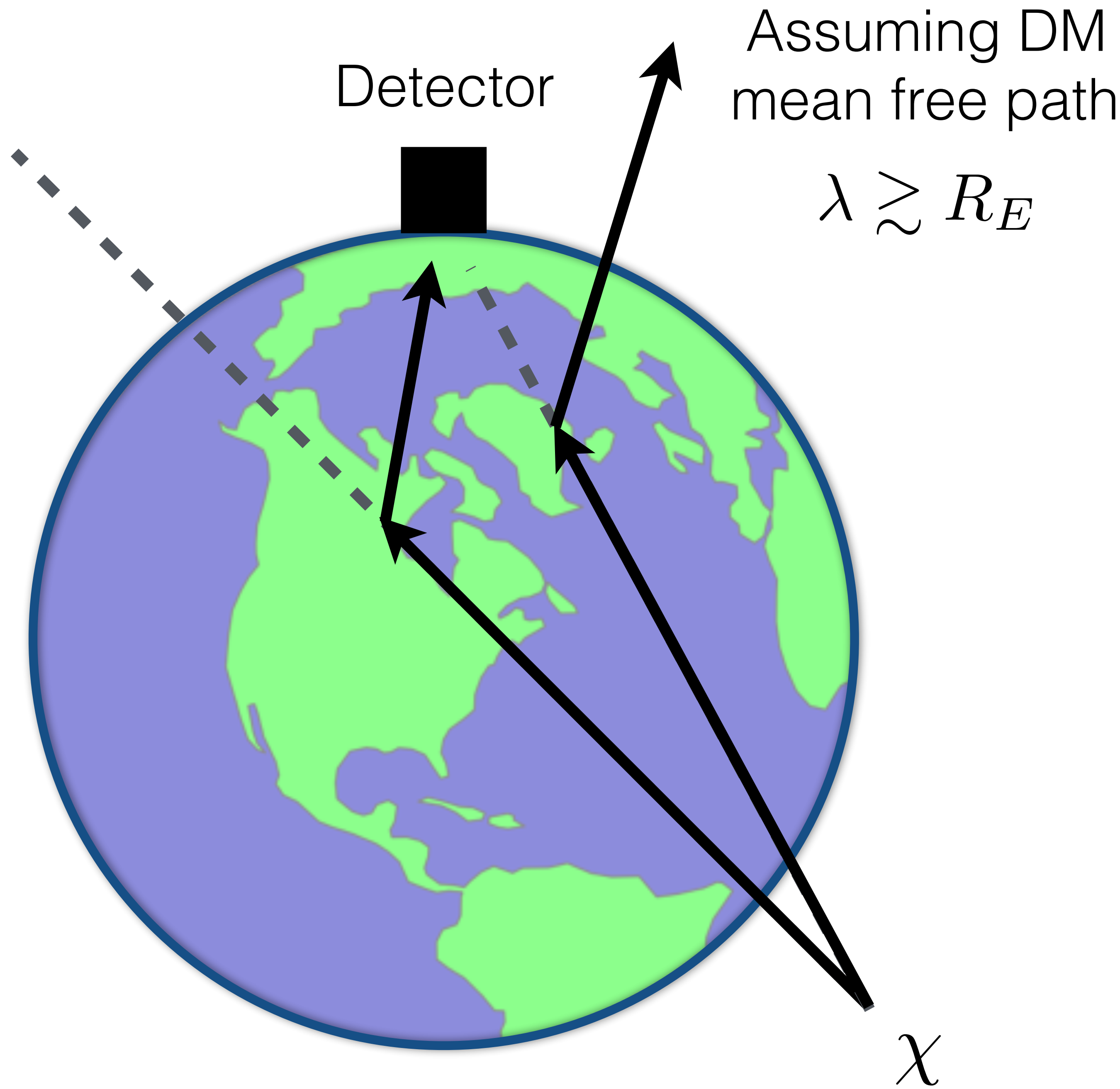
$$\lambda \gtrsim R_E$$

Can treat (without MC) in the 'single scatter' approximation…

Earth-Scattering

Consider both
attenuation **and deflection**
in an analytic framework
('Single scatter')

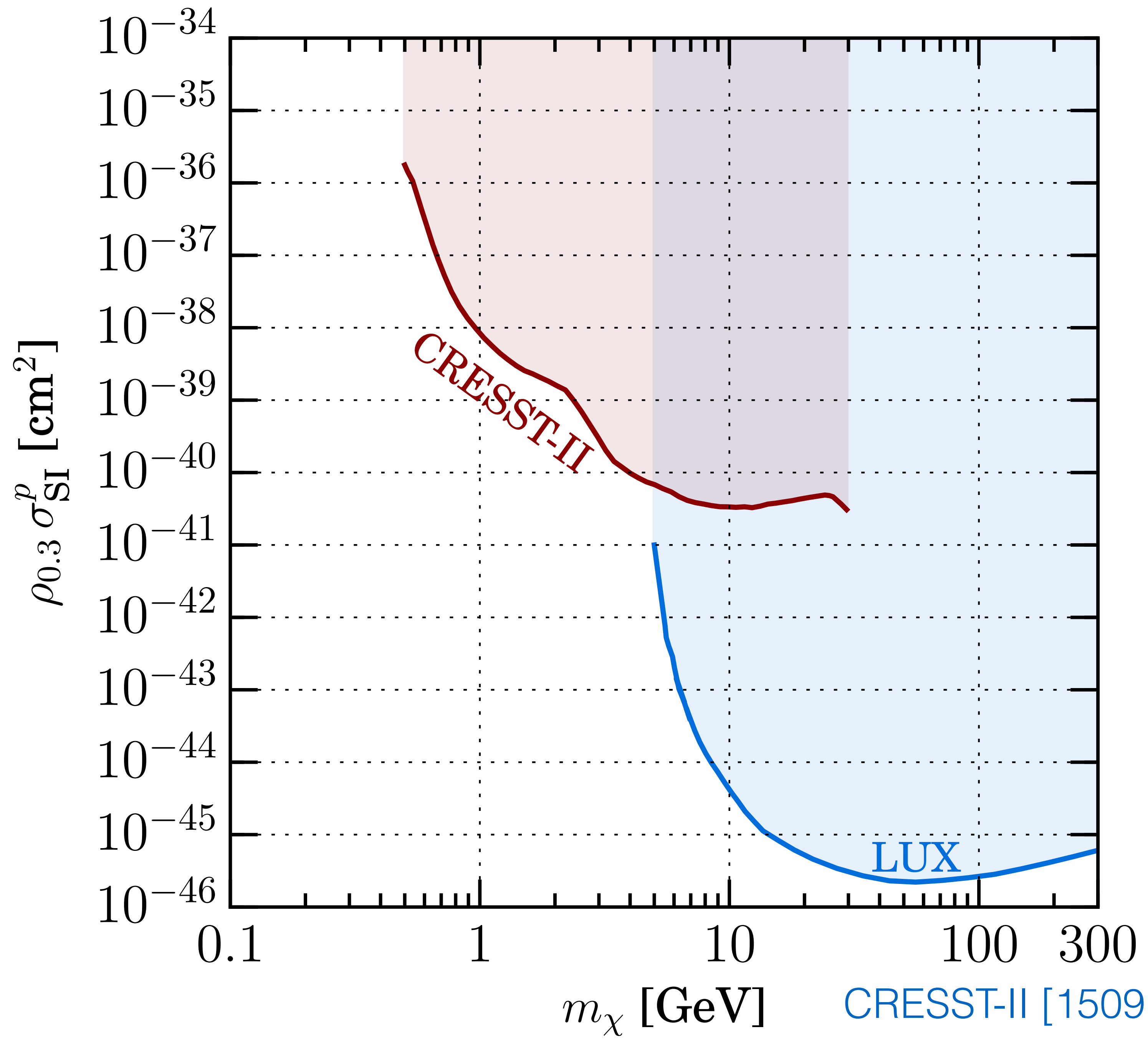
Consider **non-standard**
DM-nucleon interactions
(e.g. NREFT)



Total DM velocity distribution: $\tilde{f}(\mathbf{v}) = f_0(\mathbf{v}) - f_A(\mathbf{v}) + f_D(\mathbf{v})$

altered flux, daily modulation, directionality…

The current landscape



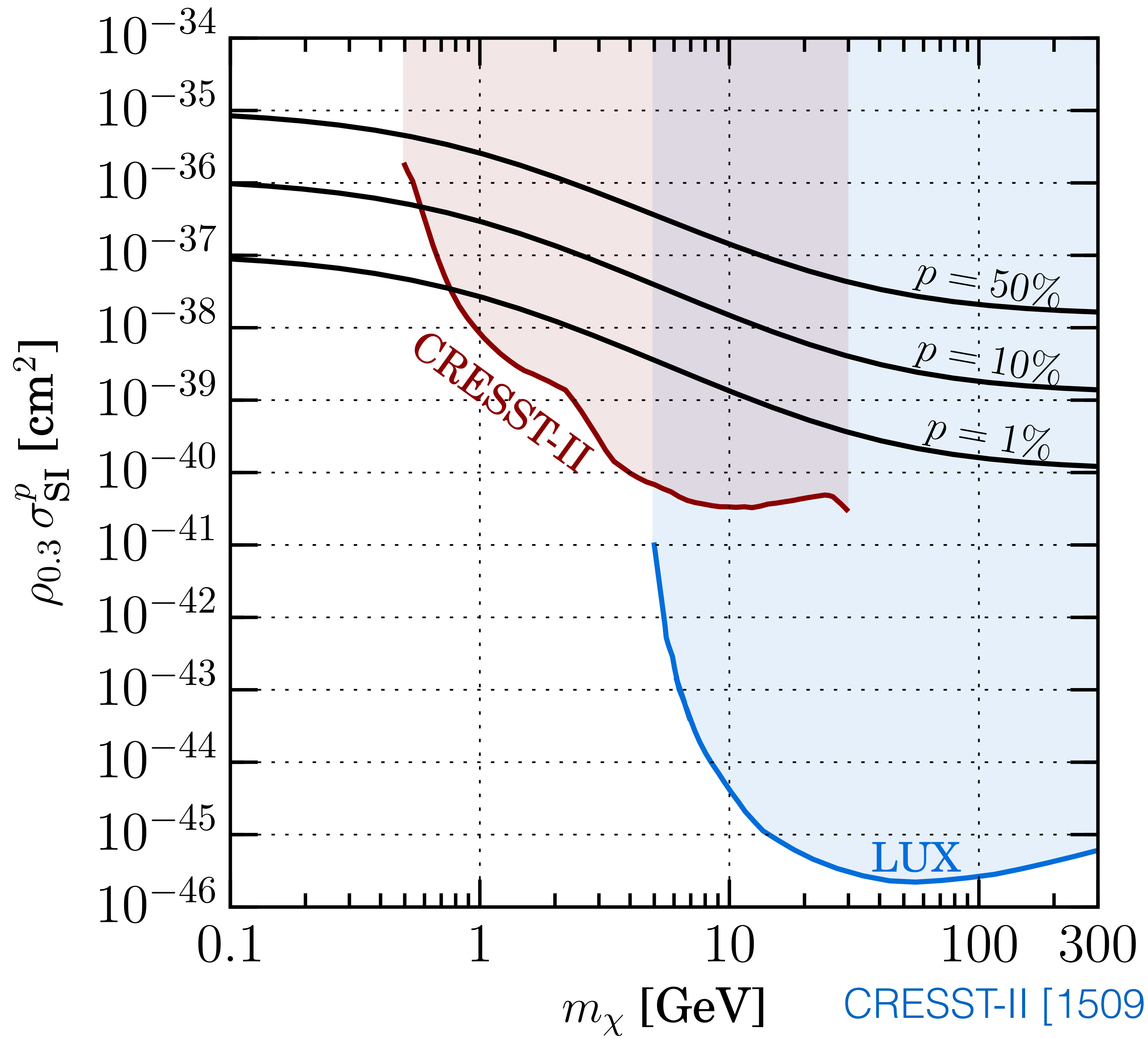
CRESST-II [1509.01515]

LUX [1608.07648]

+ many others…

How big is the probability of scattering in the Earth?

The current landscape



CRESST-II [1509.01515]

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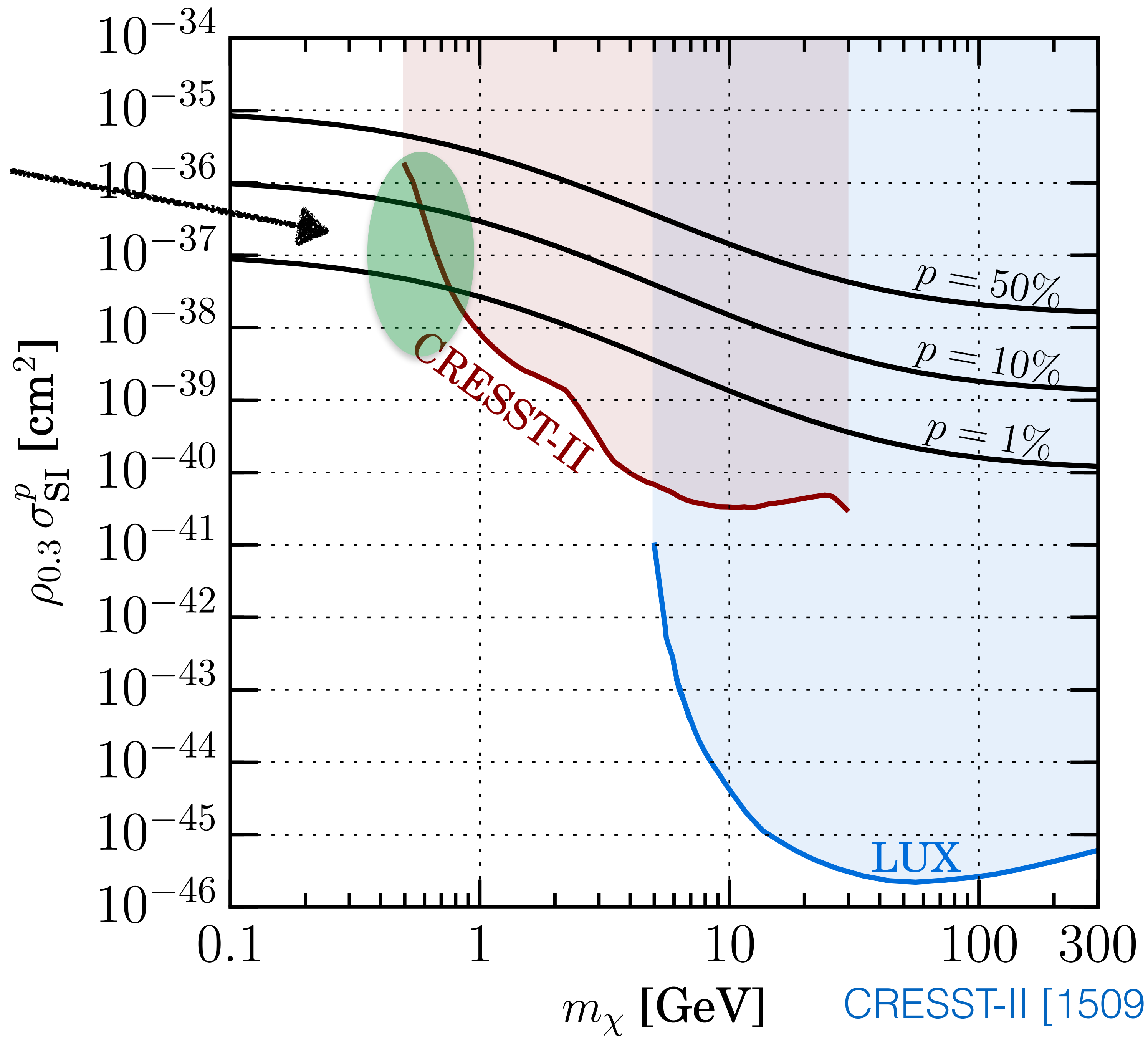
+ many others…

What effect can DM scattering in the Earth have?

The current landscape

Focus on this region



CRESST-II [1509.01515]

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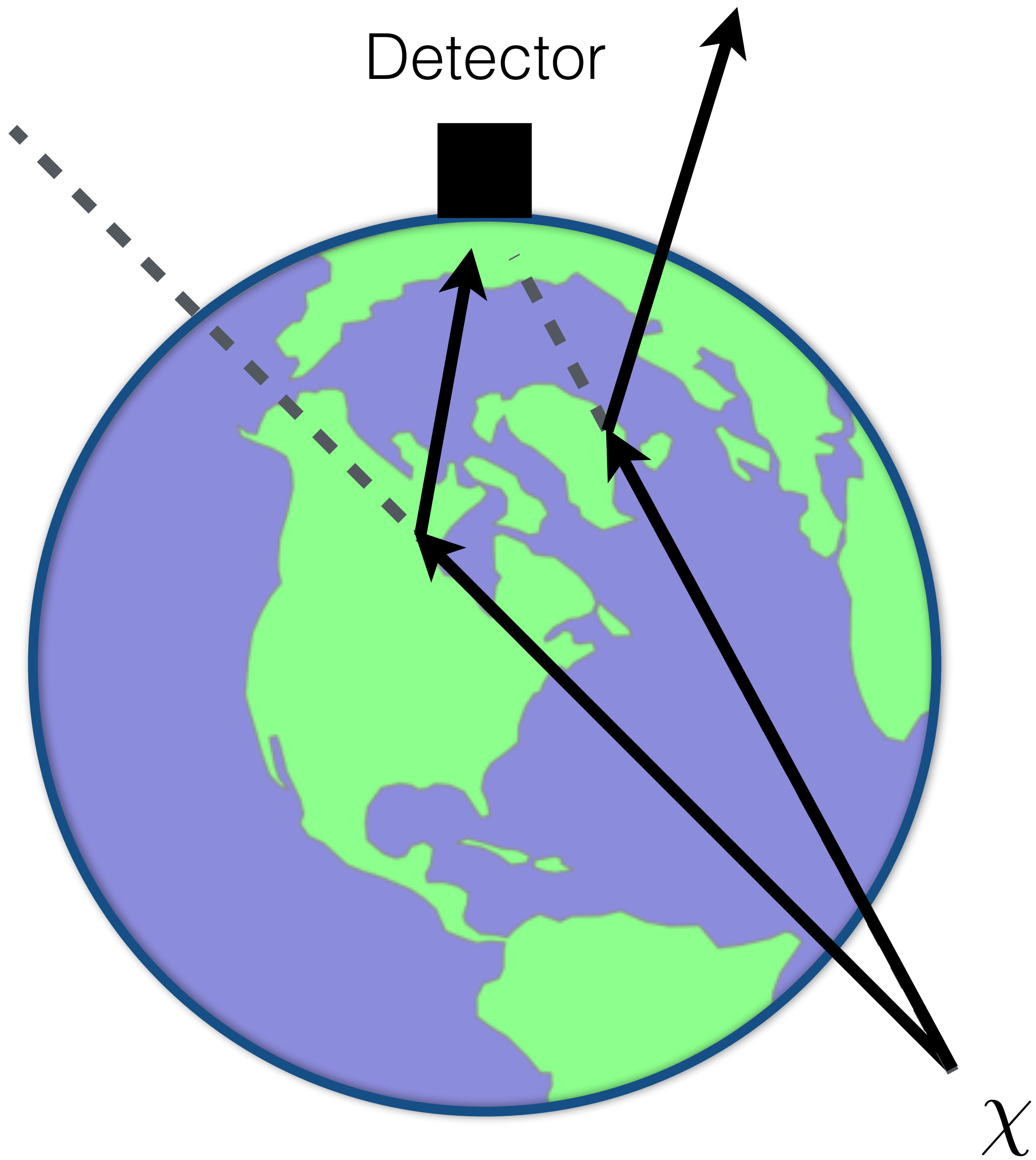
+ many others…

What effect can DM scattering in the Earth have?

Earth-Scattering Calculation



Assuming DM mean free path

$$\lambda \gtrsim R_E$$

Total DM velocity distribution: $\tilde{f}(\mathbf{v}) = f_0(\mathbf{v}) - f_A(\mathbf{v}) + f_D(\mathbf{v})$

Astrophysics of DM (the simple picture)

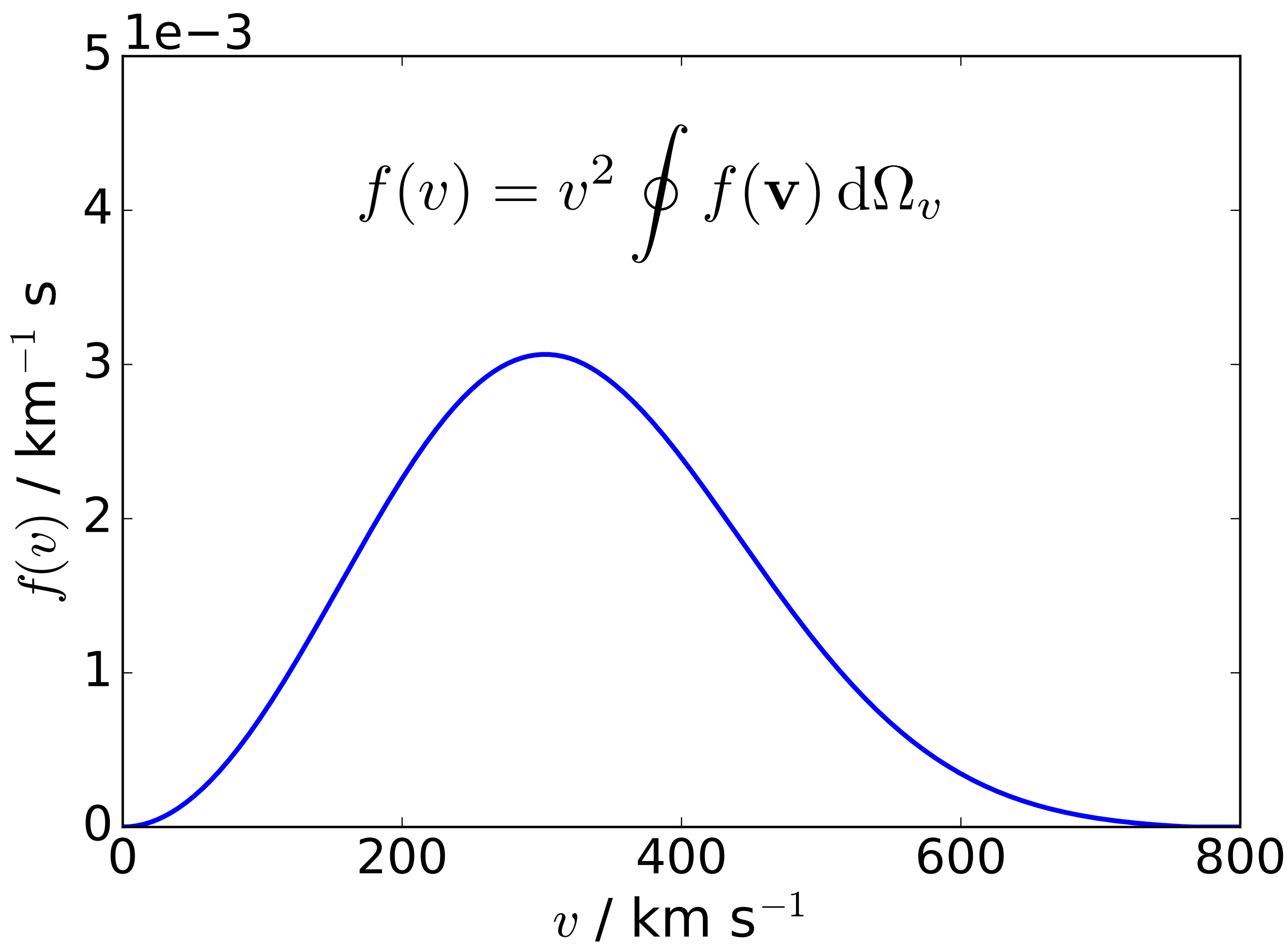
Standard Halo Model (SHM) is typically assumed: isotropic, spherically symmetric distribution of particles with $\rho(r) \propto r^{-2}$.

Leads to a Maxwell-Boltzmann (MB) distribution (*in the lab frame*):

$$f_{\mathrm{Lab}}(\mathbf{v}) = (2\pi\sigma_v^2)^{-3/2} \exp\left[-\frac{(\mathbf{v}-\mathbf{v}_\mathrm{e})^2}{2\sigma_v^2}\right] \Theta(|\mathbf{v}-\mathbf{v}_\mathrm{e}| - v_{\mathrm{esc}})$$

[See Nassim Bozorgnia's talk 06/03]

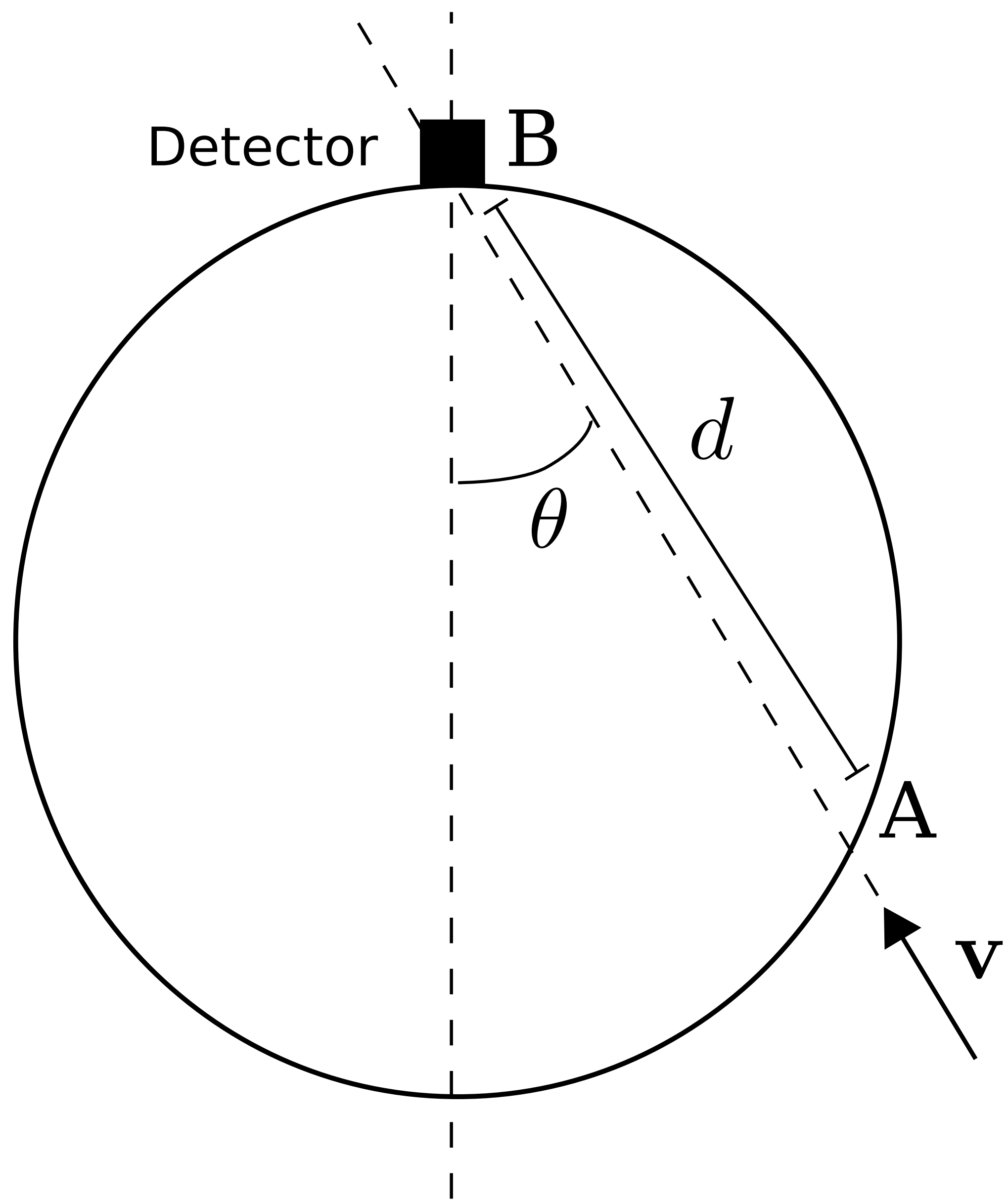
This is our 'free' distribution: $f_0(v)$



Attenuation

$\mathbf{v} = (v, \cos\theta, \phi)$

$\lambda(v)^{-1} = n\,\sigma(v)$

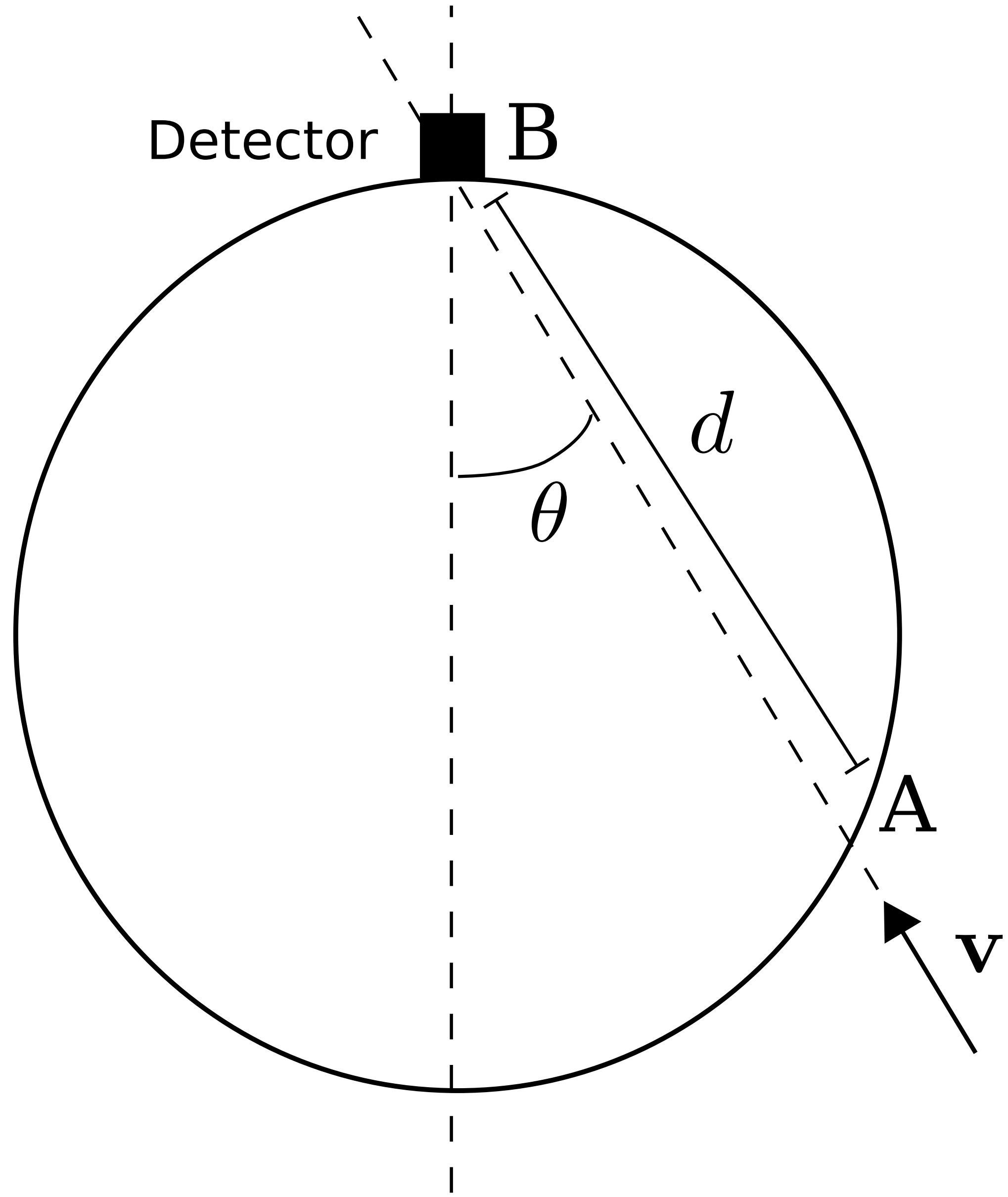


$$f_0(\mathbf{v}) - f_A(\mathbf{v}) = f_0(\mathbf{v}) \exp\left[-\frac{d(\cos\theta)}{\lambda(v)}\right]$$

Attenuation

$$\mathbf{v} = (v, \cos\theta, \phi)$$

$$\bar{\lambda}(v)^{-1} = \bar{n}\,\sigma(v)$$



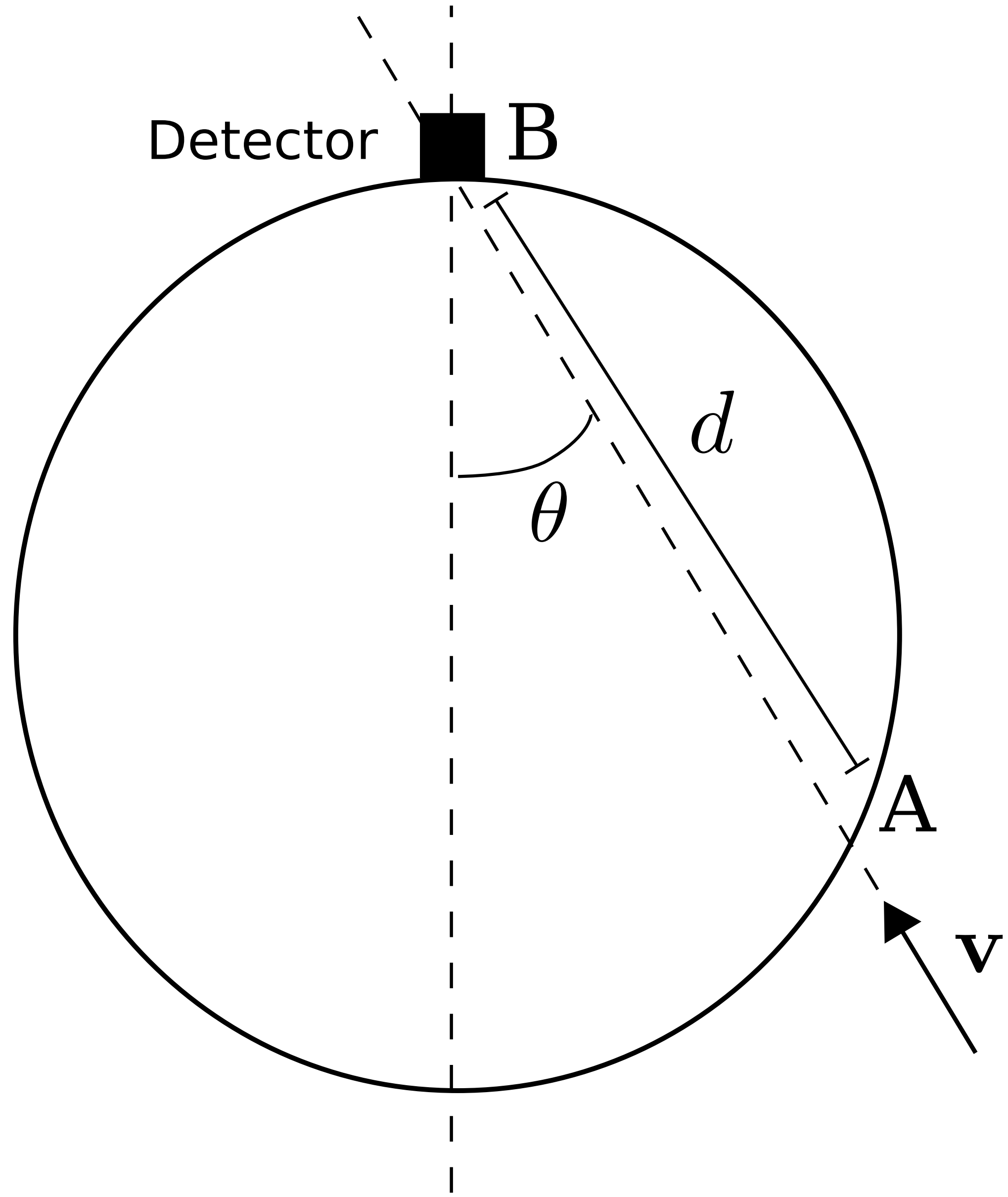
$$d_\mathrm{eff} = \frac{1}{\bar{n}}\int_\mathrm{AB} n(\mathbf{r})\mathrm{d}l$$

$$f_0(\mathbf{v}) - f_A(\mathbf{v}) = f_0(\mathbf{v})\exp\left[-\frac{d_\mathrm{eff}(\cos\theta)}{\bar{\lambda}(v)}\right]$$

Attenuation

$\mathbf{v} = (v, \cos\theta, \phi)$

$\bar{\lambda}_i(v)^{-1} = \bar{n}_i\,\sigma(v)$



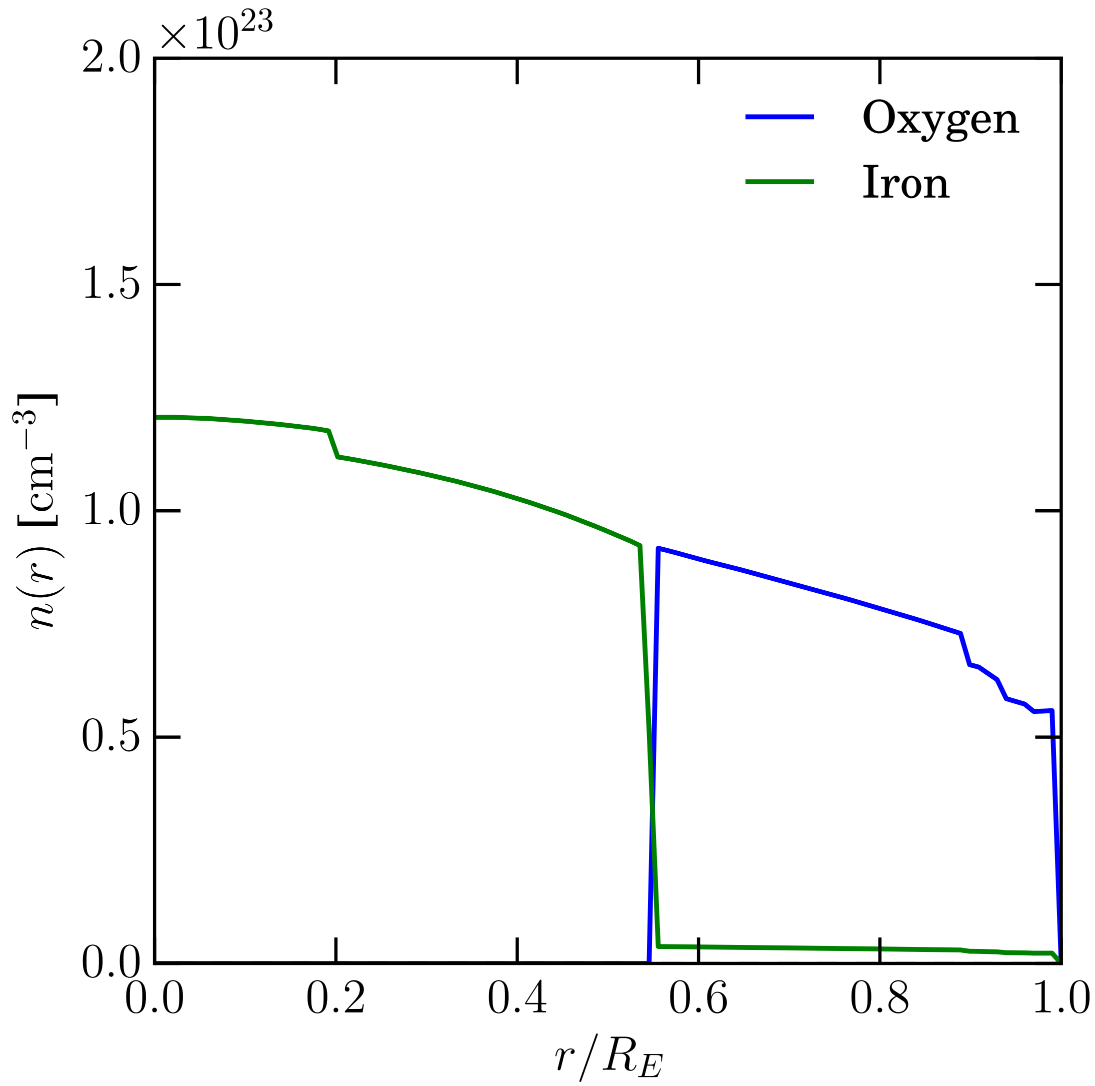
$$d_{\mathrm{eff},i} = \frac{1}{\bar{n}_i}\int_{\mathrm{AB}} n_i(\mathbf{r})\mathrm{d}l$$

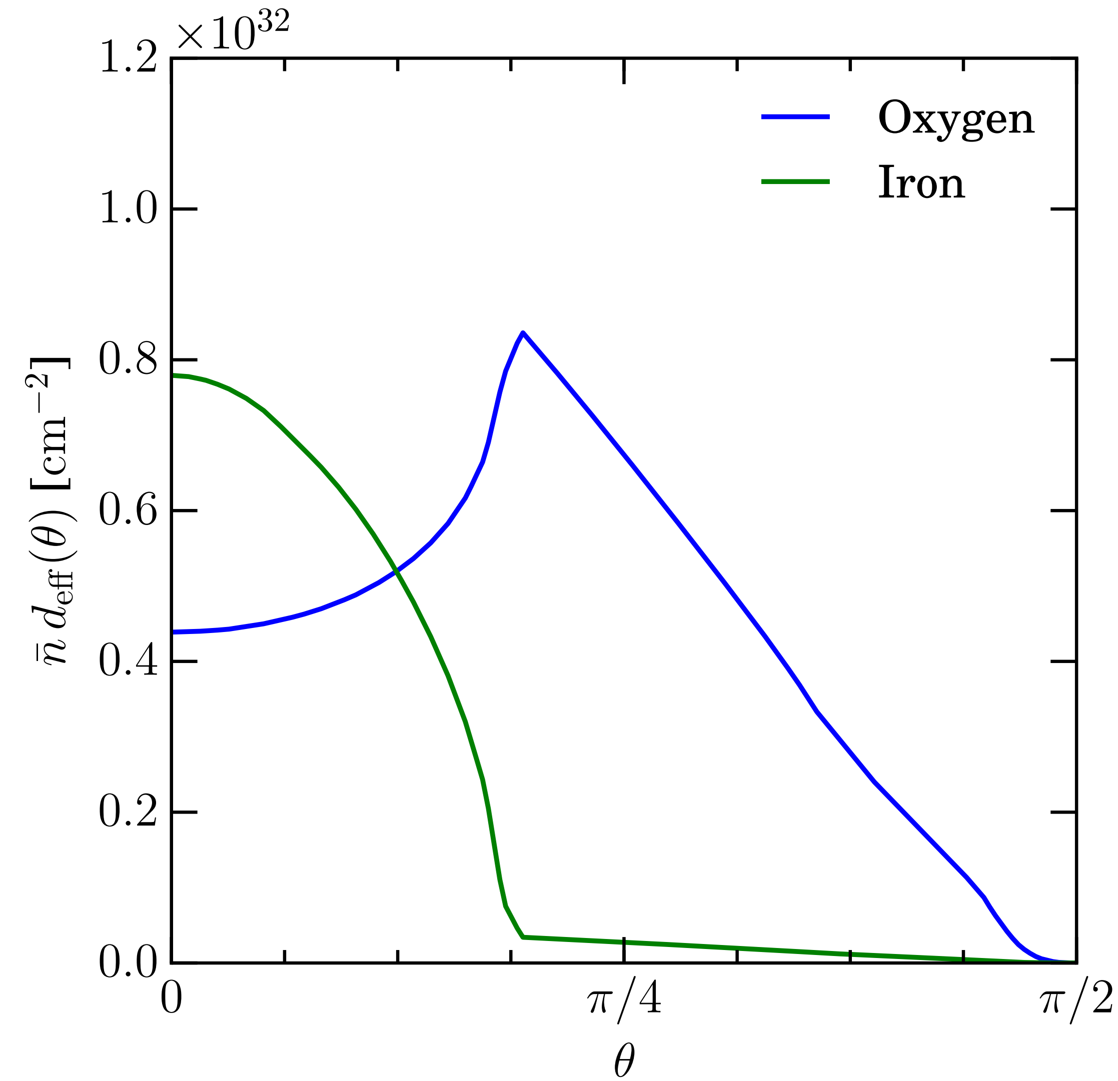
$$f_0(\mathbf{v}) - f_A(\mathbf{v}) = f_0(\mathbf{v})\exp\left[-\sum_i^{\mathrm{species}} \frac{d_{\mathrm{eff},i}(\cos\theta)}{\bar{\lambda}_i(v)}\right]$$

Sum over 8 most abundant elements in the Earth: O, Si, Mg, Fe, Ca, Na, S, Al

Effective Earth-crossing distance

Most scattering comes from Oxygen (in the mantle) and Iron (in the core)



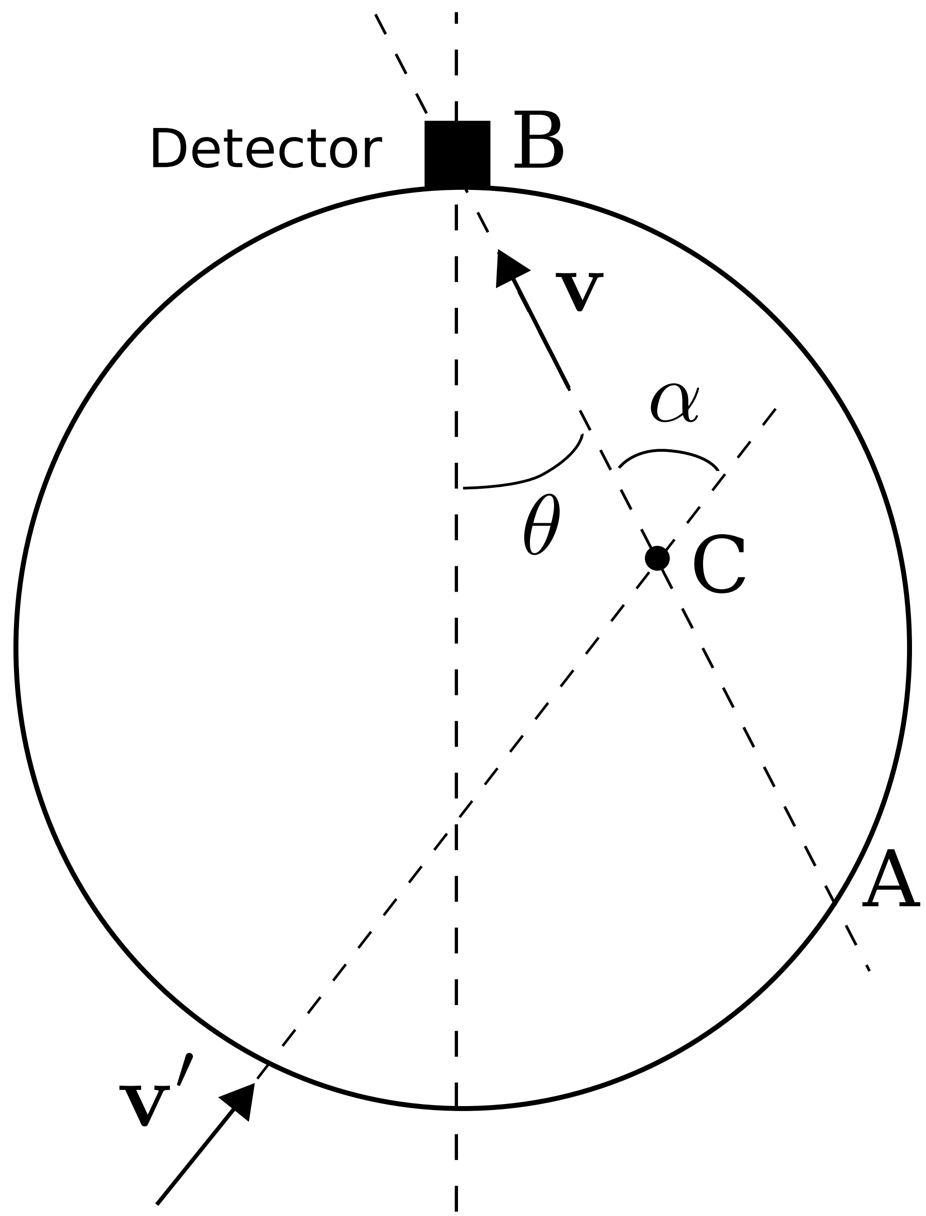


NB: little Earth-scattering for spin-dependent interactions

Deflection

$$\mathbf{v}' = (v', \cos\theta', \phi')$$
$$\mathbf{v} = (v, \cos\theta, \phi)$$

$$\bar{\lambda}_i(v)^{-1} = \bar{n}_i\,\sigma(v)$$



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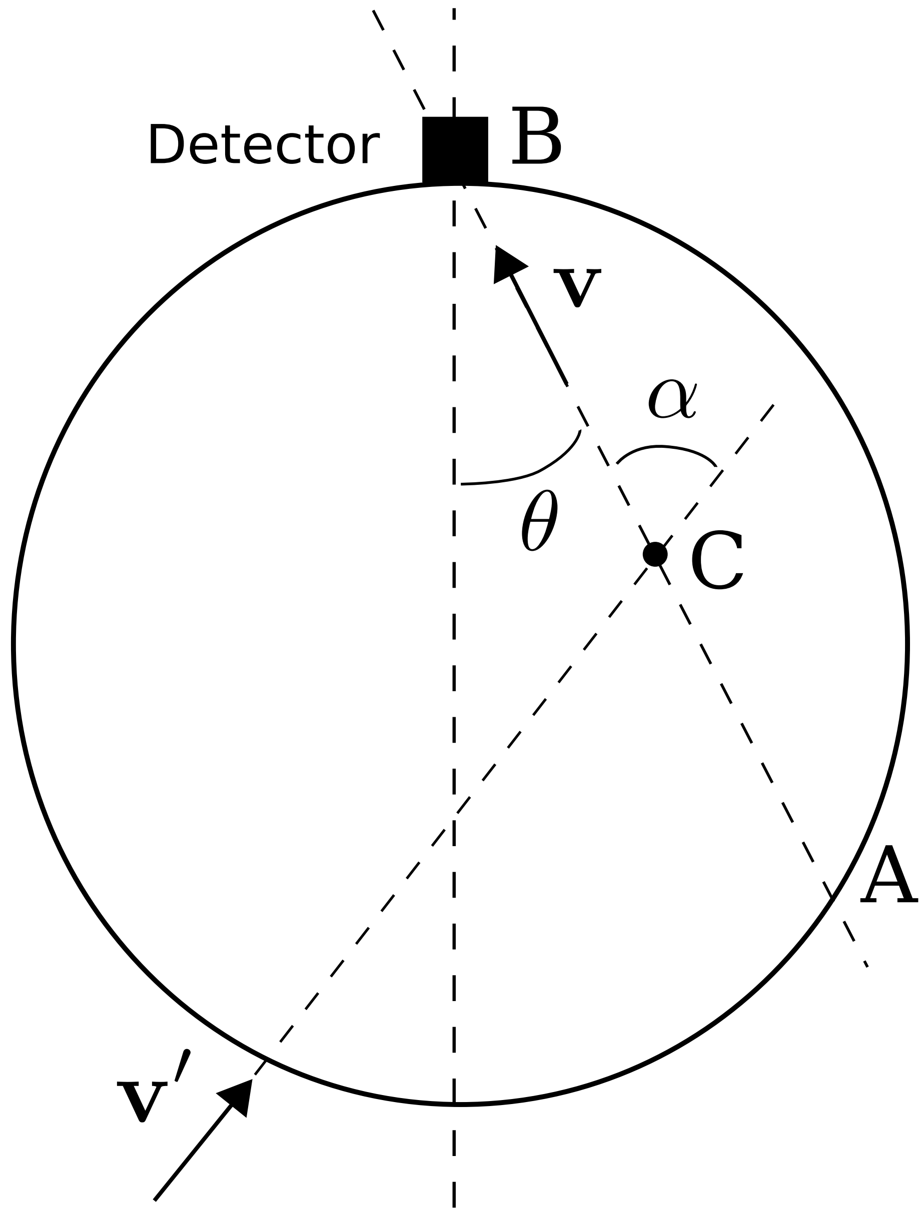

$$f_D(\mathbf{v}) = \sum_i^{\mathrm{species}} \int \mathrm{d}^2\hat{\mathbf{v}}' \,\frac{d_{\mathrm{eff},i}(\cos\theta)}{\bar{\lambda}_i(\kappa_i v)}\,\frac{(\kappa_i)^4}{2\pi} f_0(\kappa_i v, \hat{\mathbf{v}}')P_i(\cos\alpha)$$

[Detailed calculation in the paper]

$$\kappa_i = v'/v$$

Deflection

$$\mathbf{v}' = (v', \cos\theta', \phi')$$
$$\mathbf{v} = (v, \cos\theta, \phi)$$

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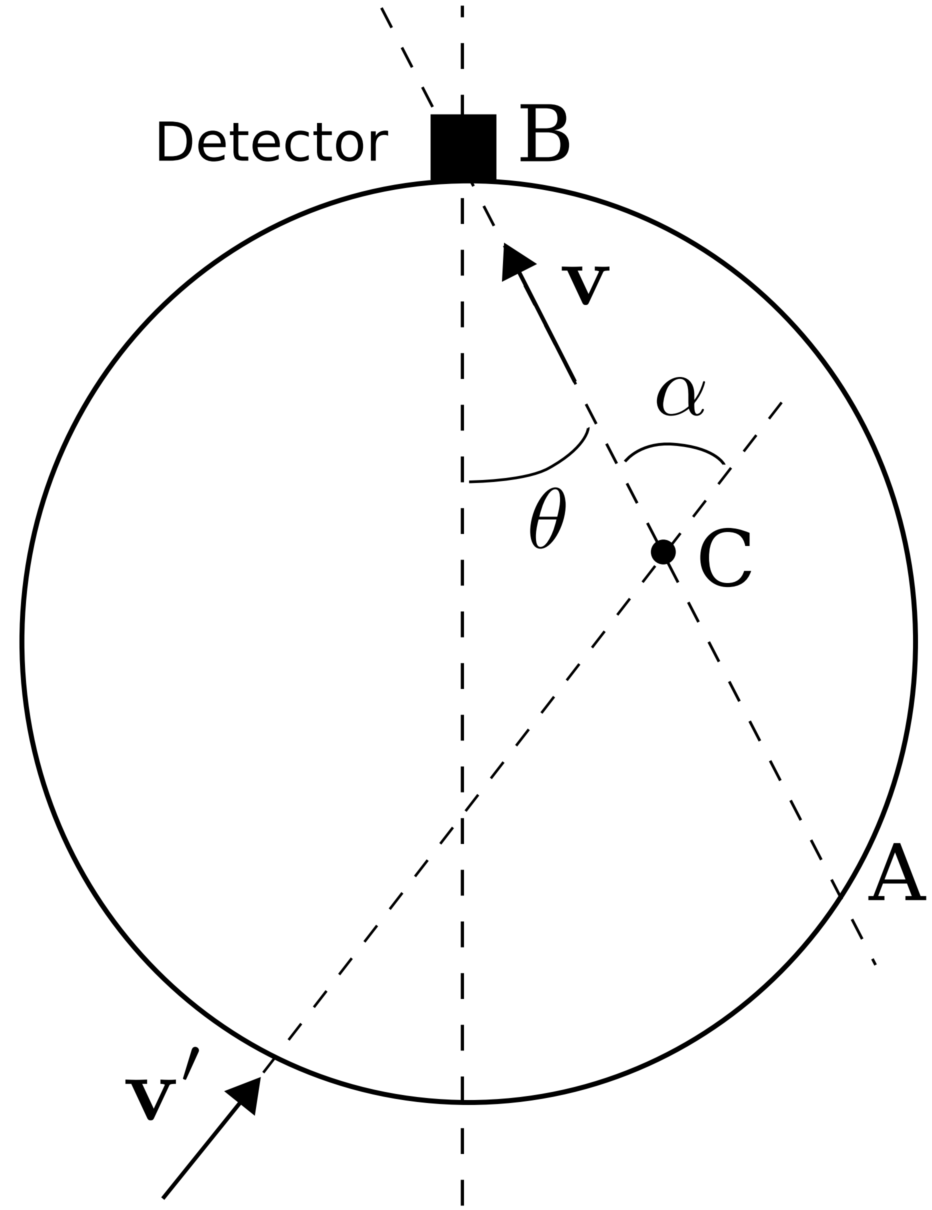

Depends on differential cross section

$$f_D(\mathbf{v}) = \sum_i^{\text{species}} \int \mathrm{d}^2\hat{\mathbf{v}}'\, \frac{d_{\mathrm{eff},i}(\cos\theta)}{\boxed{\bar{\lambda}_i(\kappa_i v)}} \frac{(\kappa_i)^4}{2\pi} f_0(\kappa_i v, \hat{\mathbf{v}}') \boxed{P_i(\cos\alpha)}$$

Depends on total cross section

$$\kappa_i = v'/v$$

NREFT operator basis

Write down all possible non-relativistic (NR) WIMP-*nucleon* operators which can mediate the *elastic* scattering.

[Fan et al - 1008.1591, Fitzpatrick et al. - 1203.3542]

SI $\boxed{\mathcal{O}_1 = 1}$

SD $\boxed{\mathcal{O}_4 = \vec{S}_\chi \cdot \vec{S}_N}$

[1008.1591, 1203.3542, 1308.6288, 1505.03117]

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SI

SD

$$\boxed{\mathcal{O}_1 = 1}$$

$$\mathcal{O}_3 = i\vec{S}_N \cdot (\vec{q} \times \vec{v}^\perp)/m_N$$

$$\boxed{\mathcal{O}_4 = \vec{S}_\chi \cdot \vec{S}_N}$$

$$\mathcal{O}_5 = i\vec{S}_\chi \cdot (\vec{q} \times \vec{v}^\perp)/m_N$$

$$\mathcal{O}_6 = (\vec{S}_\chi \cdot \vec{q})(\vec{S}_N \cdot \vec{q})/m_N^2$$

$$\mathcal{O}_7 = \vec{S}_N \cdot \vec{v}^\perp$$

$$\mathcal{O}_8 = \vec{S}_\chi \cdot \vec{v}^\perp$$

$$\mathcal{O}_9 = i\vec{S}_\chi \cdot (\vec{S}_N \times \vec{q})/m_N$$

$$\mathcal{O}_{10} = i\vec{S}_N \cdot \vec{q}/m_N$$

$$\mathcal{O}_{11} = i\vec{S}_\chi \cdot \vec{q}/m_N$$

$$\mathcal{O}_{12} = \vec{S}_\chi \cdot (\vec{S}_N \times \vec{v}^\perp)$$

$$\mathcal{O}_{13} = i(\vec{S}_\chi \cdot \vec{v}^\perp)(\vec{S}_N \cdot \vec{q})/m_N$$

$$\mathcal{O}_{14} = i(\vec{S}_\chi \cdot \vec{q})(\vec{S}_N \cdot \vec{v}^\perp)/m_N$$

$$\mathcal{O}_{15} = -(\vec{S}_\chi \cdot \vec{q})((\vec{S}_N \times \vec{v}^\perp) \cdot \vec{q}/m_N^2$$

$$\vdots$$

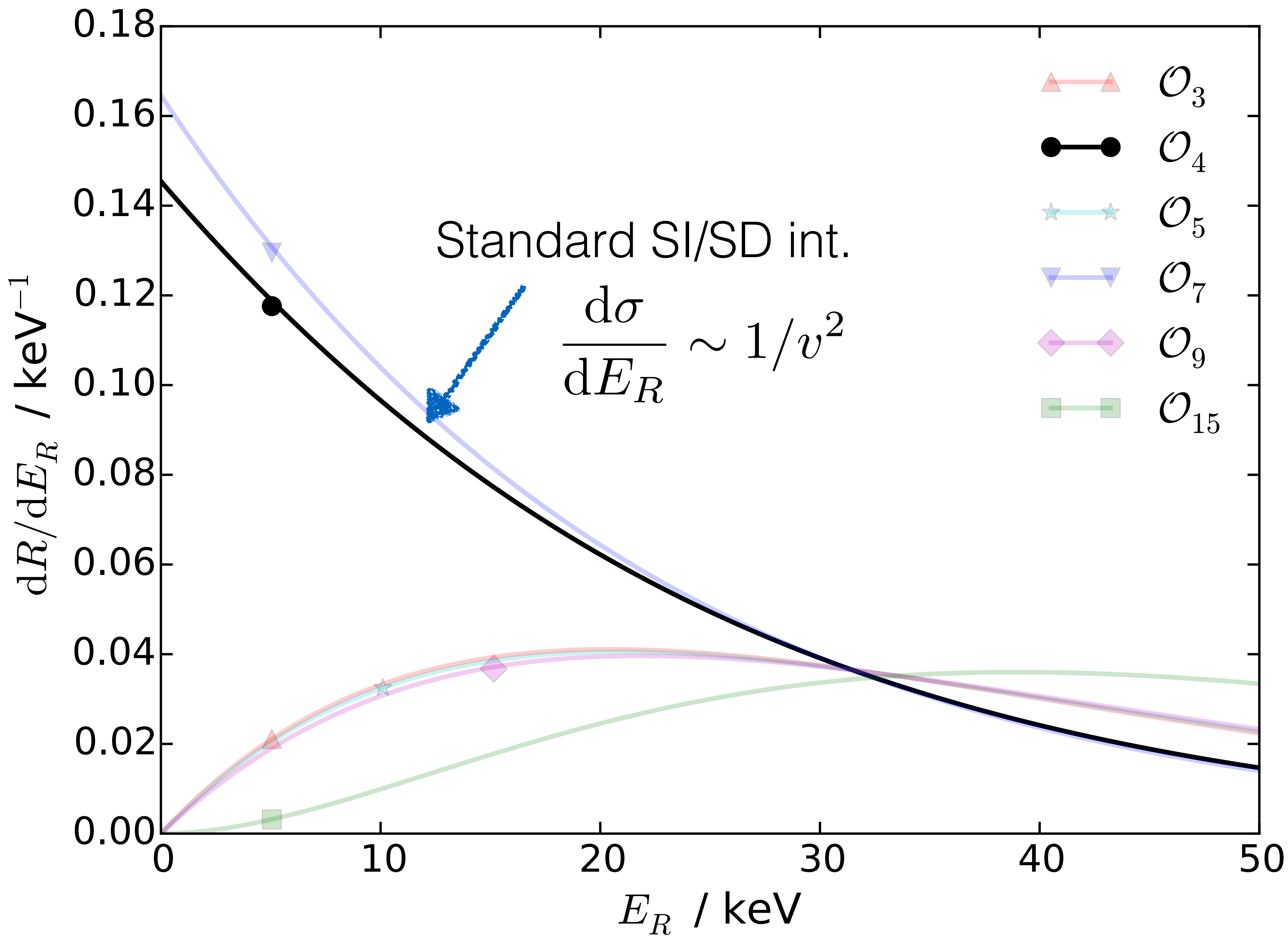
NB: two sets of operators, one for protons and one for neutrons…

$$\boxed{\vec{v}_\perp = \vec{v} + \frac{\vec{q}}{2\mu_{\chi N}}}$$

[1008.1591, 1203.3542, 1308.6288, 1505.03117]

Energy spectra

$m_\chi = 100\ \mathrm{GeV}$



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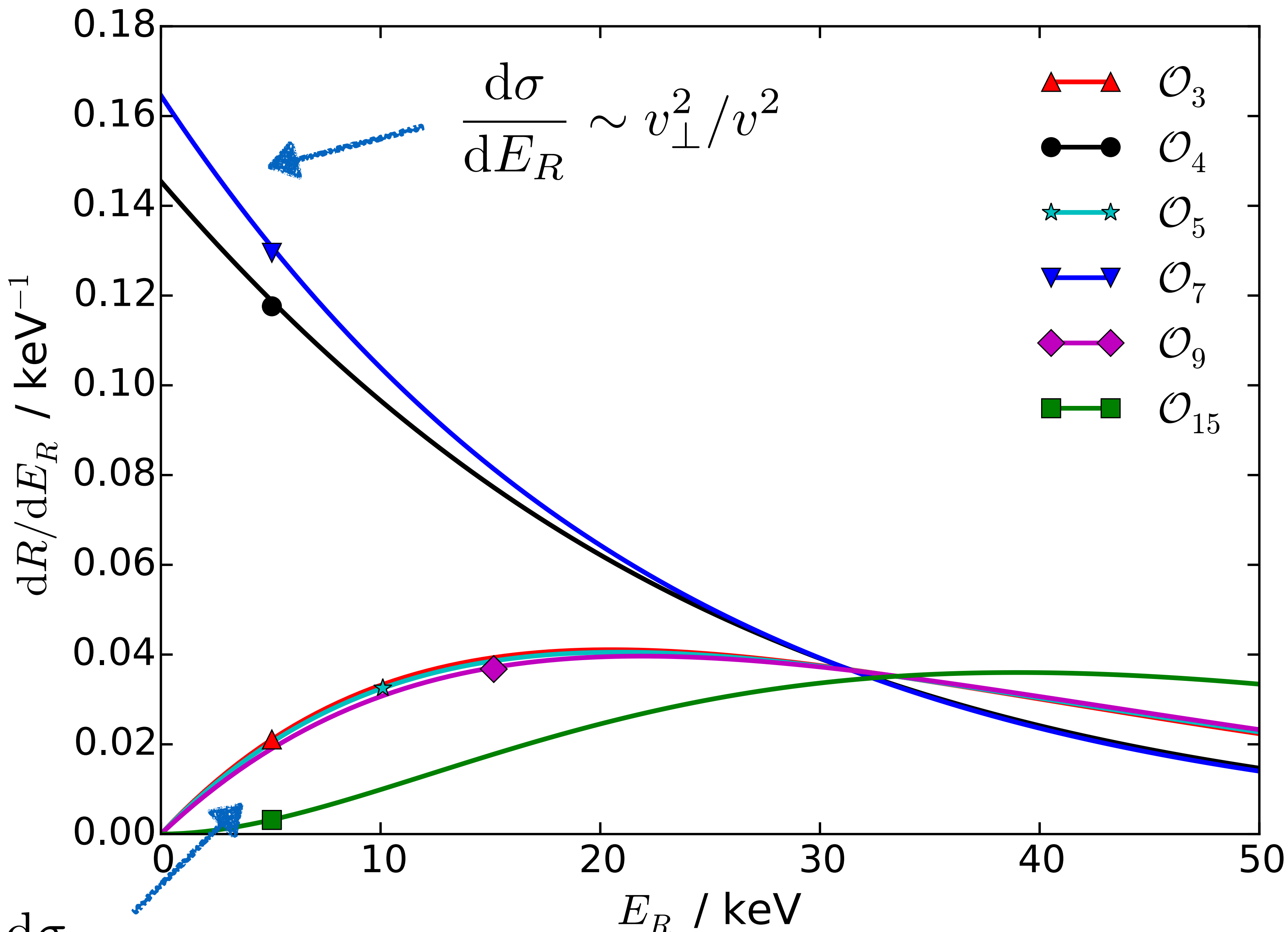
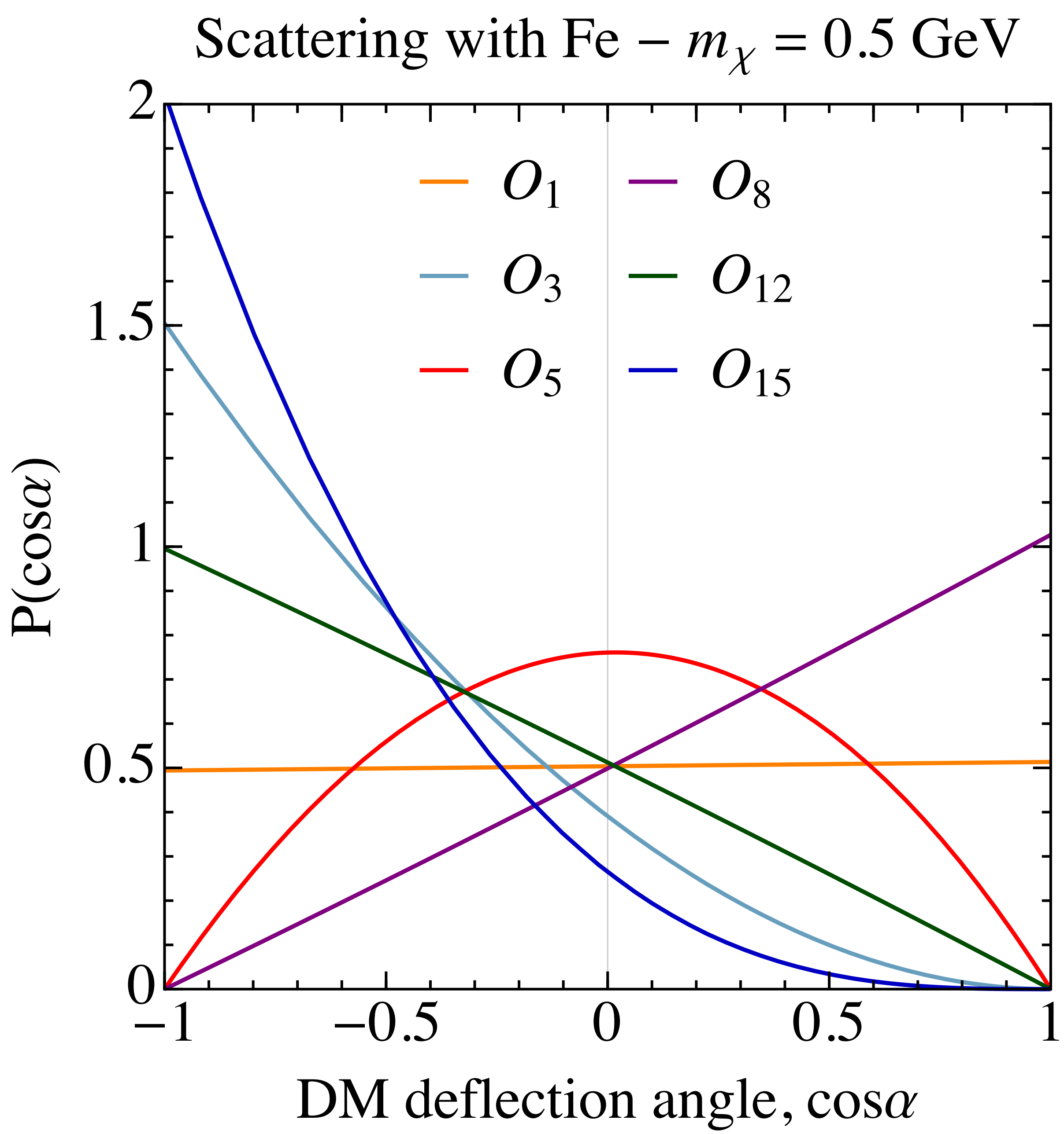


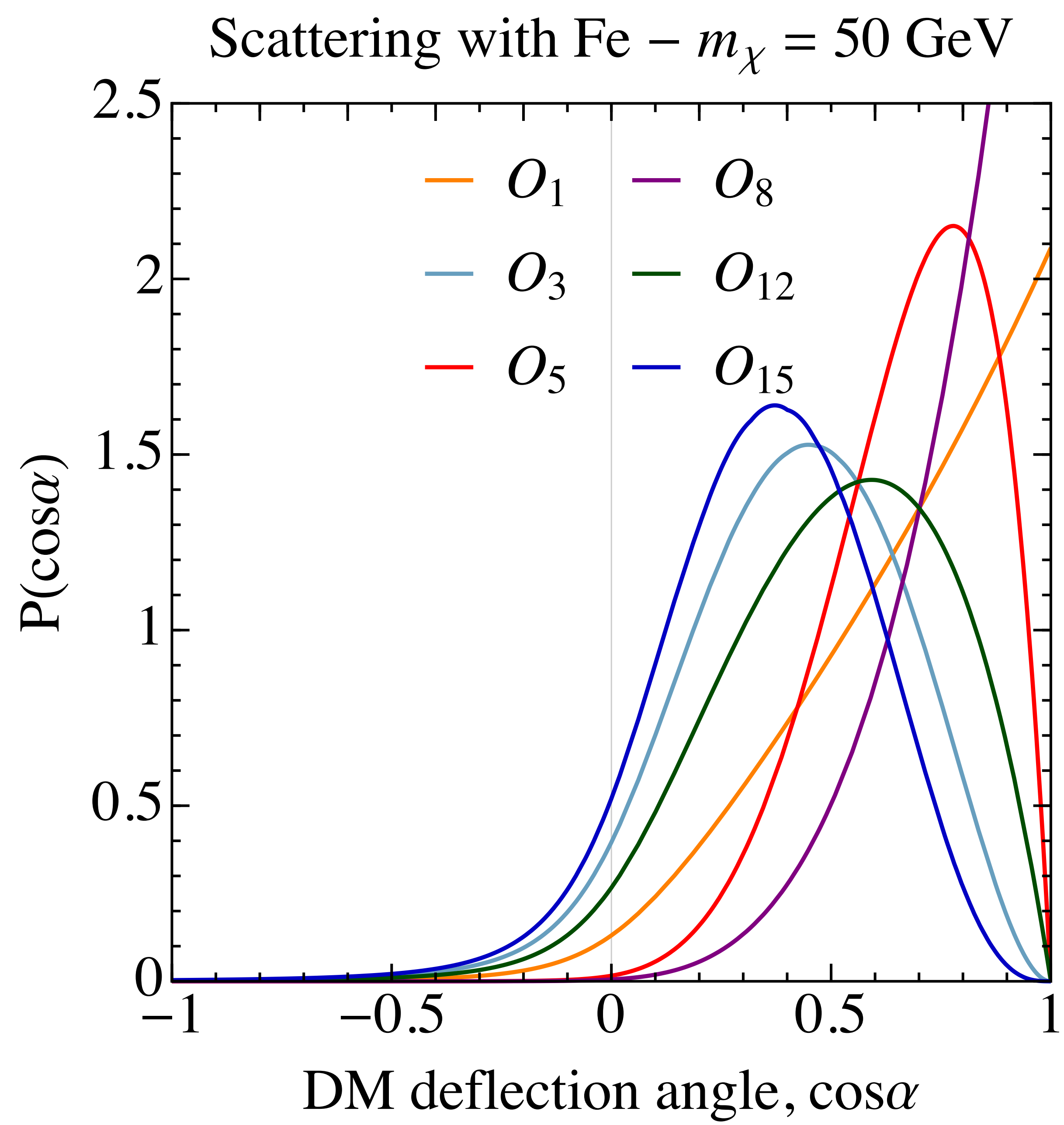
$$\frac{\mathrm{d}\sigma}{\mathrm{d}E_R} \sim v_\perp^2/v^2$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}E_R} \sim q^2/v^2$$

DM deflection distribution

$$P(\cos\alpha) = \frac{1}{\sigma}\frac{\mathrm{d}\sigma}{\mathrm{d}E_R}\frac{\mathrm{d}E_R}{\mathrm{d}\cos\alpha}$$



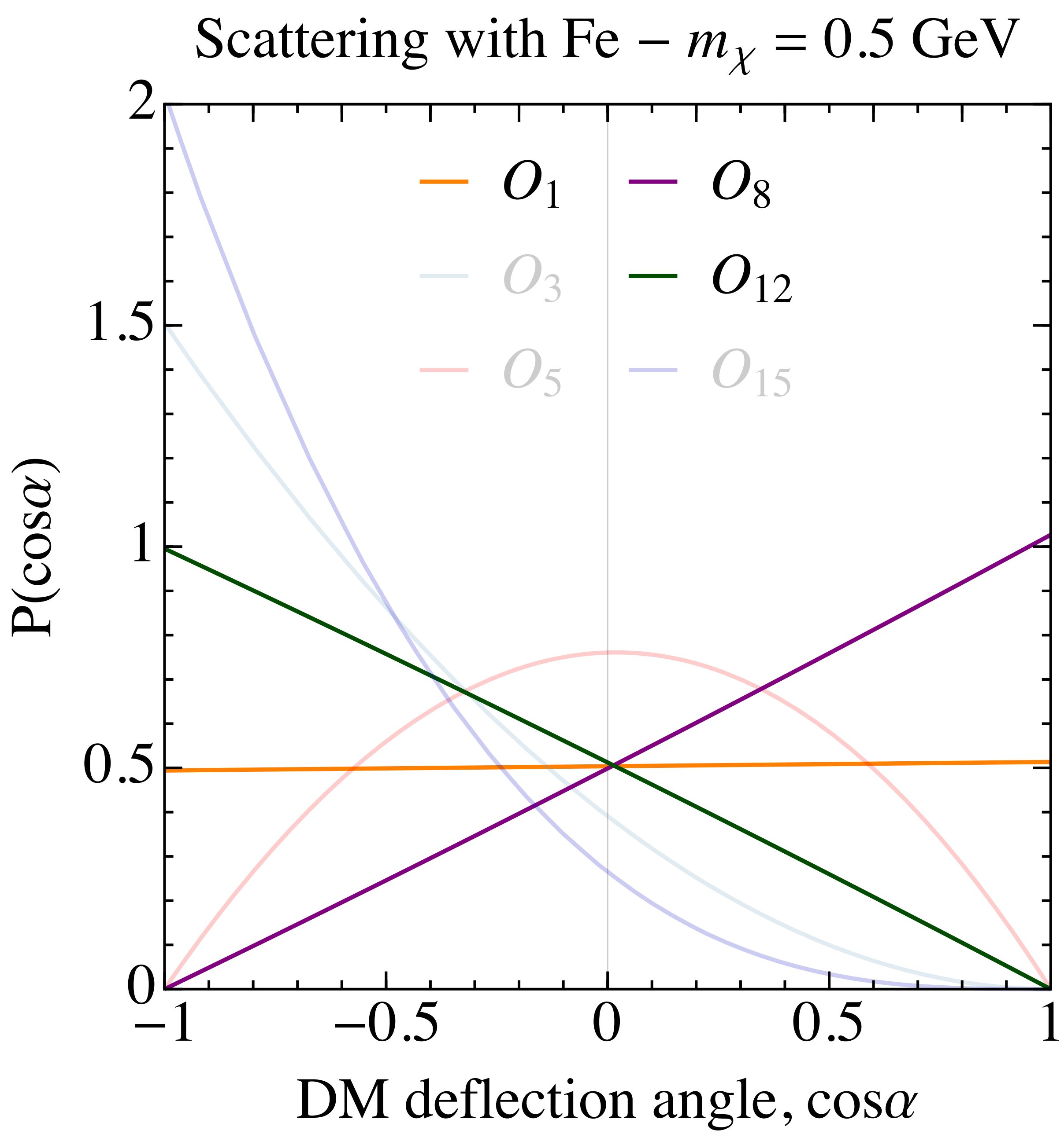


Backward

Forward

DM deflection distribution

$$P(\cos\alpha) = \frac{1}{\sigma}\frac{\mathrm{d}\sigma}{\mathrm{d}E_R}\frac{\mathrm{d}E_R}{\mathrm{d}\cos\alpha}$$



Backward

Forward

Standard SI

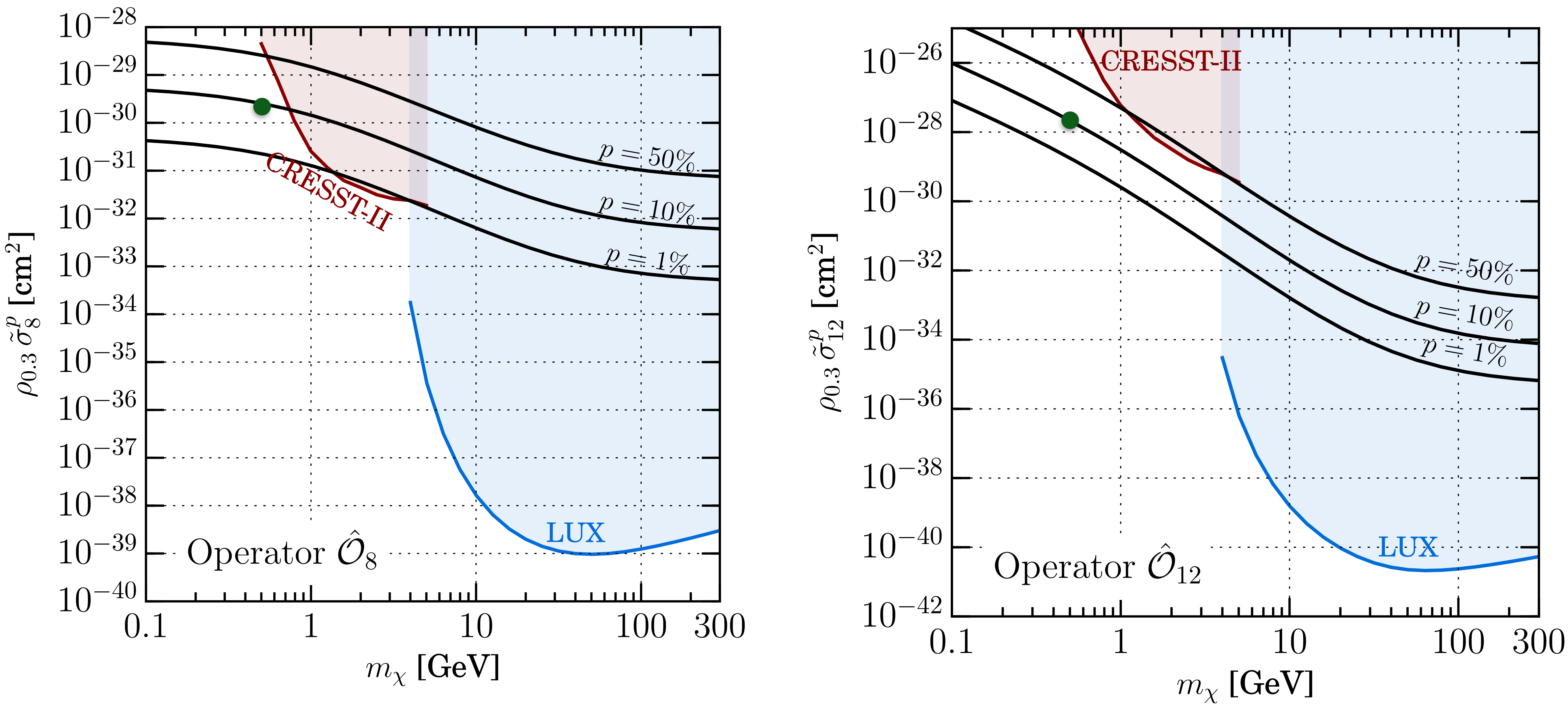
$$\mathcal{O}_1 = \mathbb{1} \Rightarrow \frac{\mathrm{d}\sigma}{\mathrm{d}E_R} \sim \frac{1}{v^2}$$

$$\mathcal{O}_8 = \vec{S}_\chi \cdot \vec{v}^\perp \Rightarrow \frac{\mathrm{d}\sigma}{\mathrm{d}E_R} \sim (1 - \frac{m_\mathcal{N} E_R}{2\mu_{\chi\mathcal{N}}^2 v^2})$$

$$\mathcal{O}_{12} = \vec{S}_\chi \cdot (\vec{S}_N \times \vec{v}^\perp) \Rightarrow \frac{\mathrm{d}\sigma}{\mathrm{d}E_R} \sim \frac{E_R}{v^2}$$

Constraints on NREFT operators

Focus on SI operator (O_1), as well as O_8 and O_{12}:



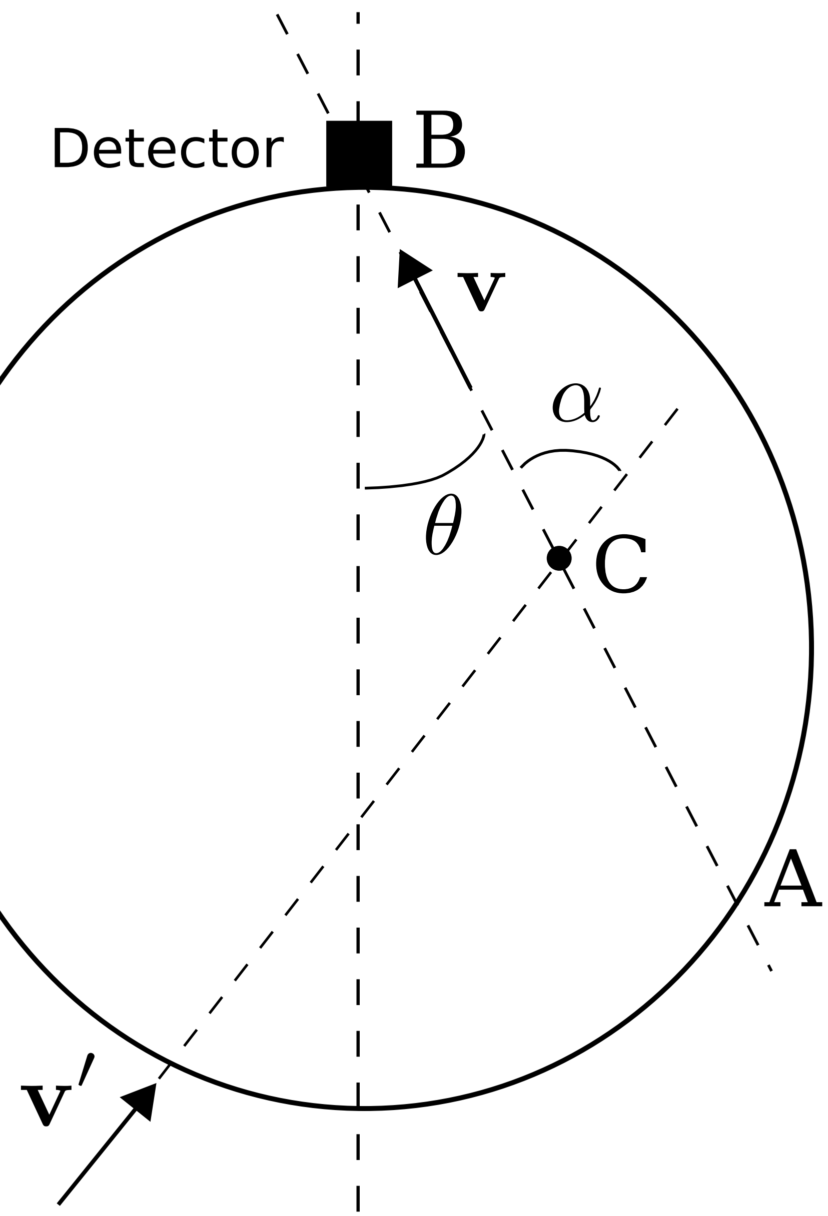
Focus on low mass DM: $m_\chi = 0.5$ GeV

Fix couplings to give 10% probability of scattering

Deflection

$$\mathbf{v}' = (v', \cos\theta', \phi')$$
$$\mathbf{v} = (v, \cos\theta, \phi)$$

$$\bar{\lambda}_i(v)^{-1} = \bar{n}_i\,\sigma(v)$$



Now we have everything we need!

Depends on differential cross section

$$f_D(\mathbf{v}) = \sum_i^{\text{species}} \int \mathrm{d}^2\hat{\mathbf{v}}'\, \frac{d_{\mathrm{eff},i}(\cos\theta)}{\boxed{\bar{\lambda}_i(\kappa_i v)}} \frac{(\kappa_i)^4}{2\pi} f_0(\kappa_i v, \hat{\mathbf{v}}') \boxed{P_i(\cos\alpha)}$$

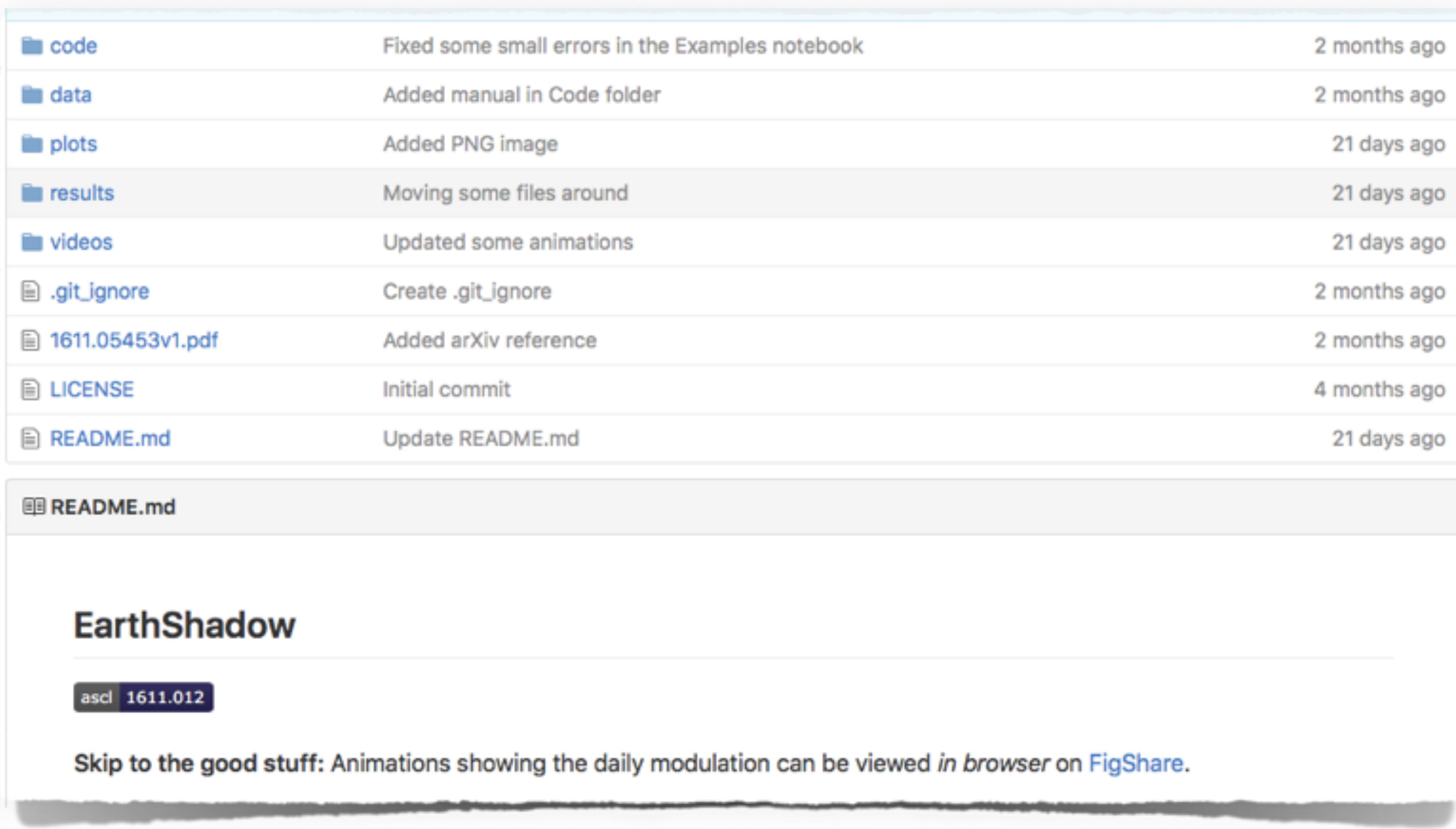
Depends on total cross section

$$\kappa_i = v'/v$$

EARTHSHADOW Code

EARTHSHADOW code is available online at:
github.com/bradkav/EarthShadow

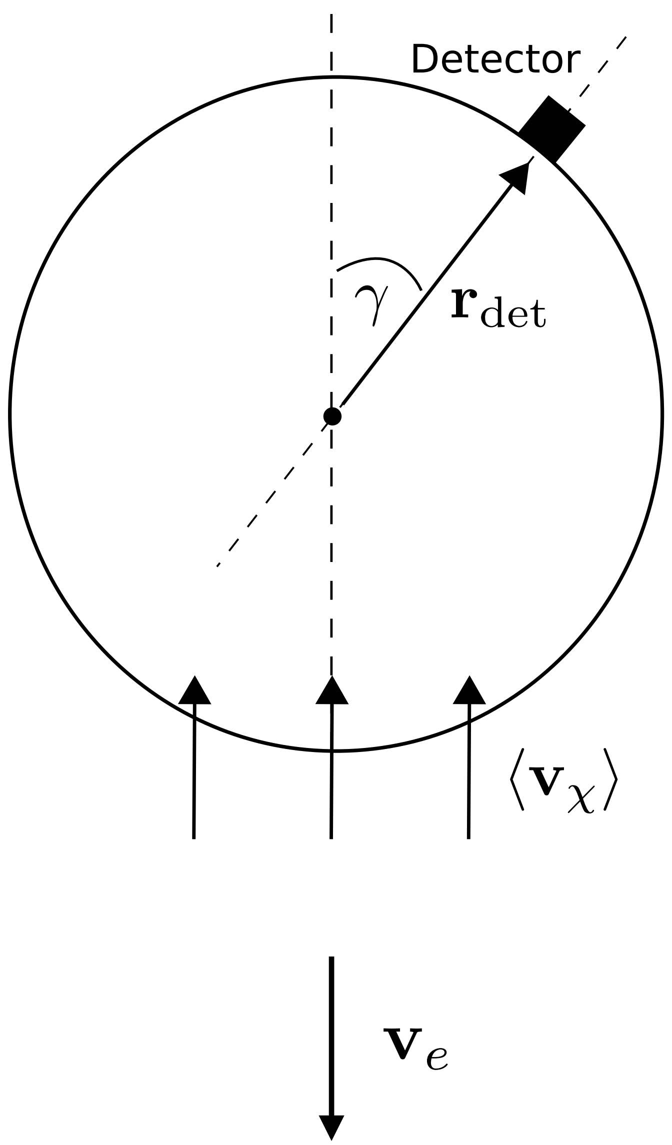
Including routines, numerical results, plots and animations…

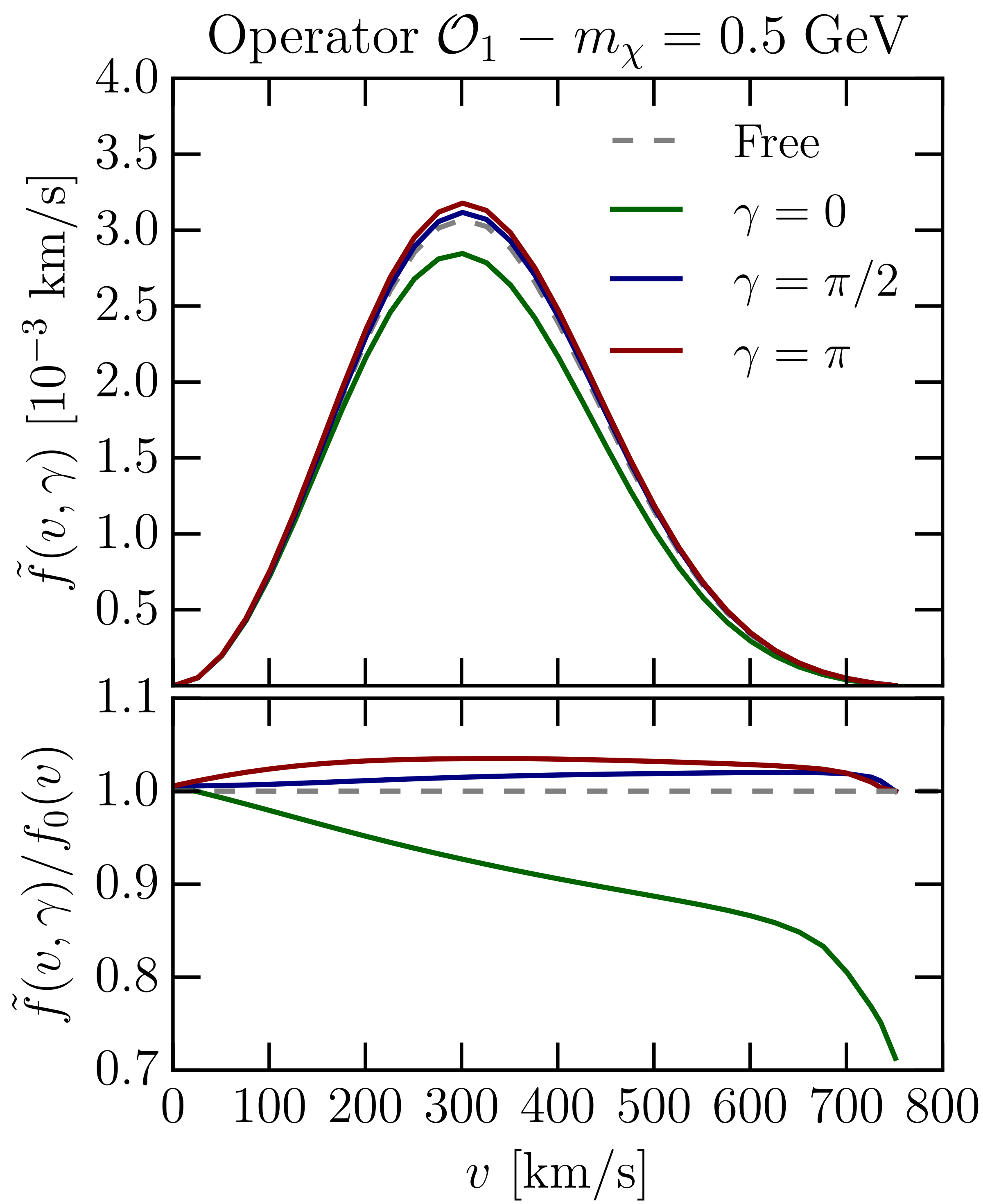


Results

Speed Distribution - Operator 1

Calculate DM speed distribution after Earth scattering: $\tilde{f}(v, \gamma)$





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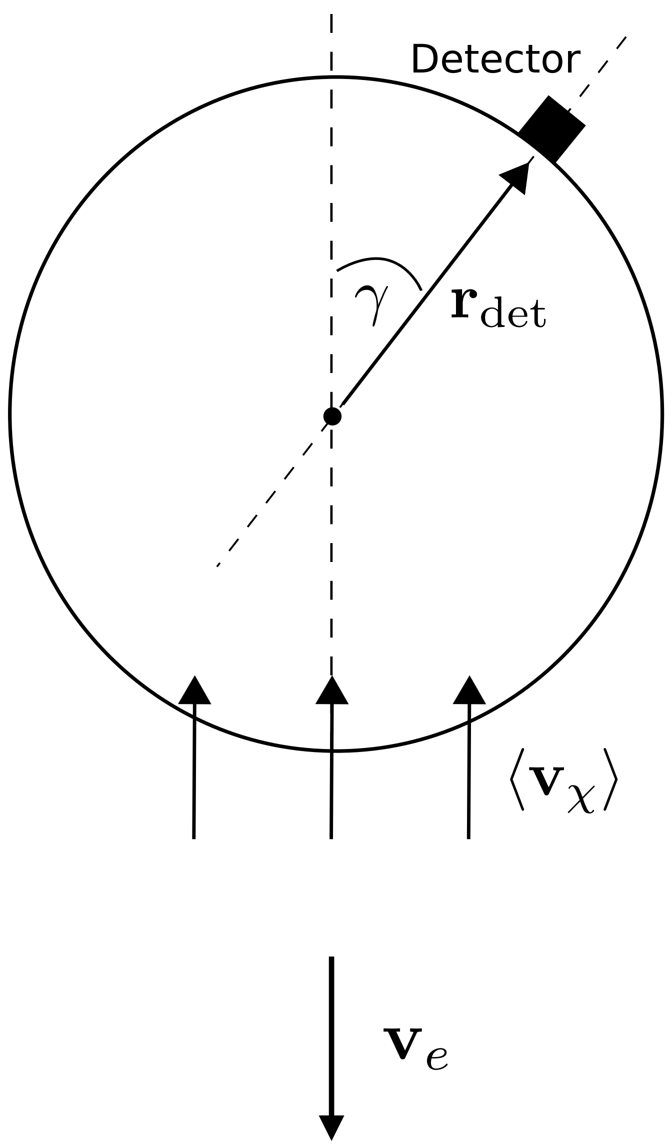
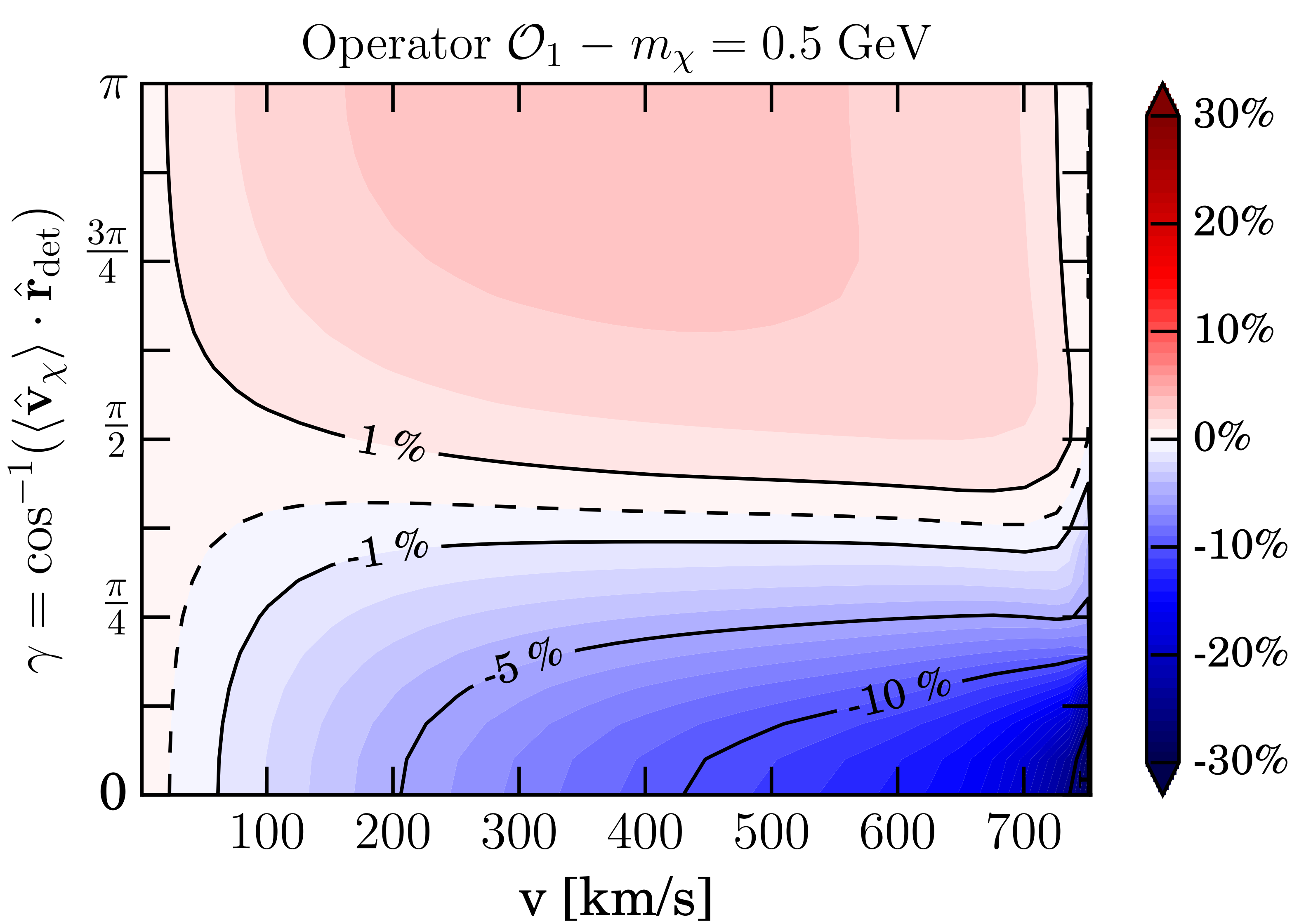
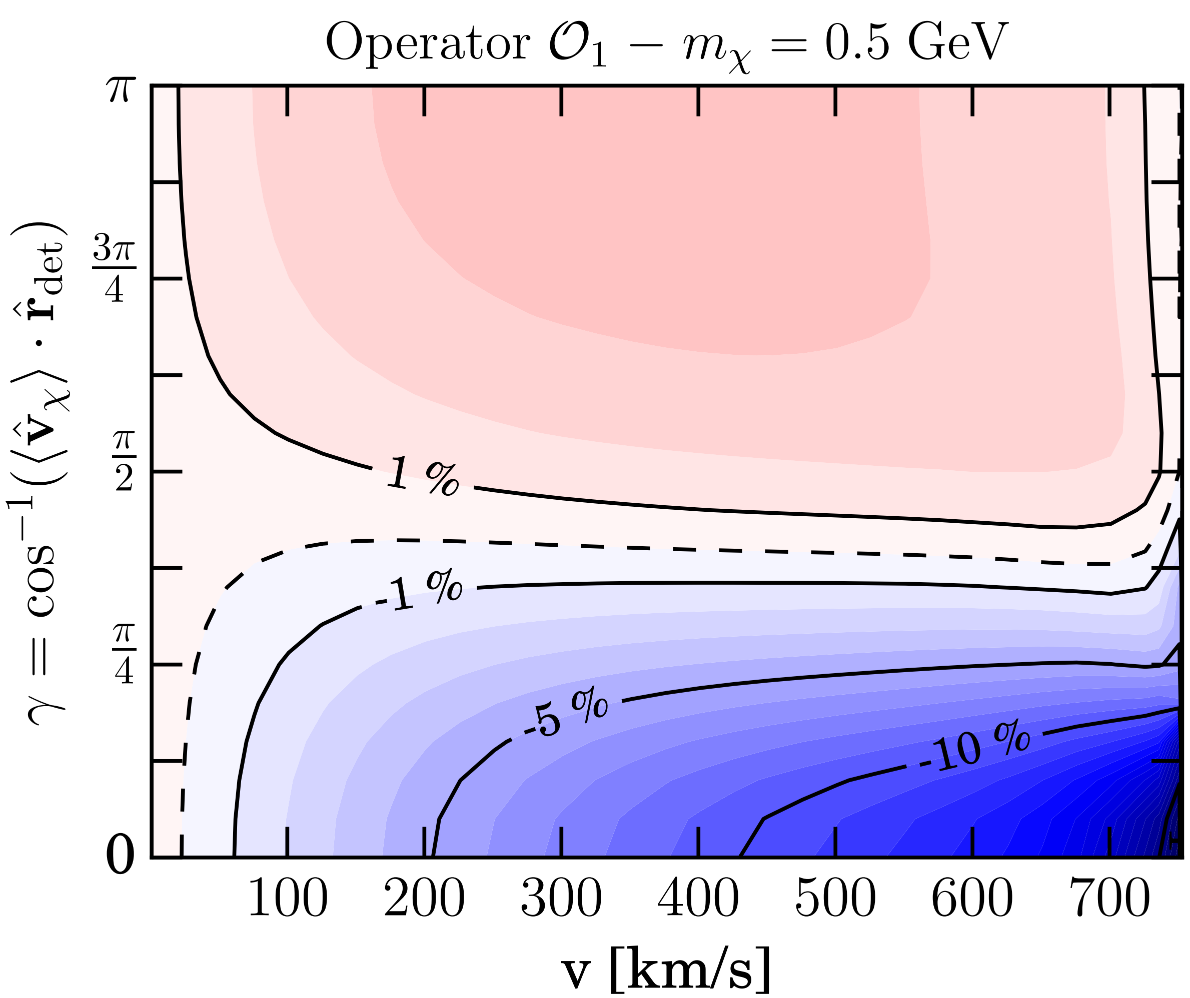
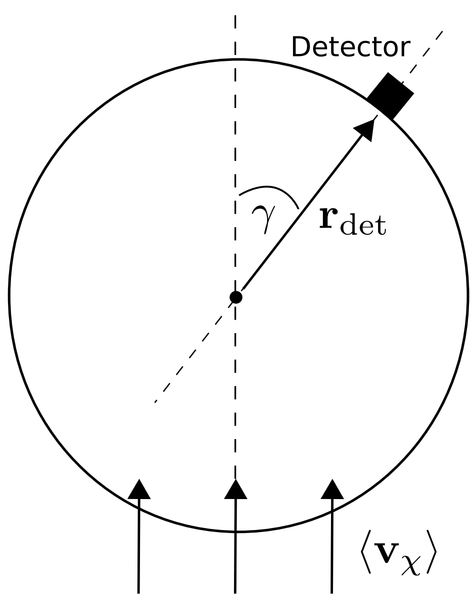


Percentage change in speed dist.

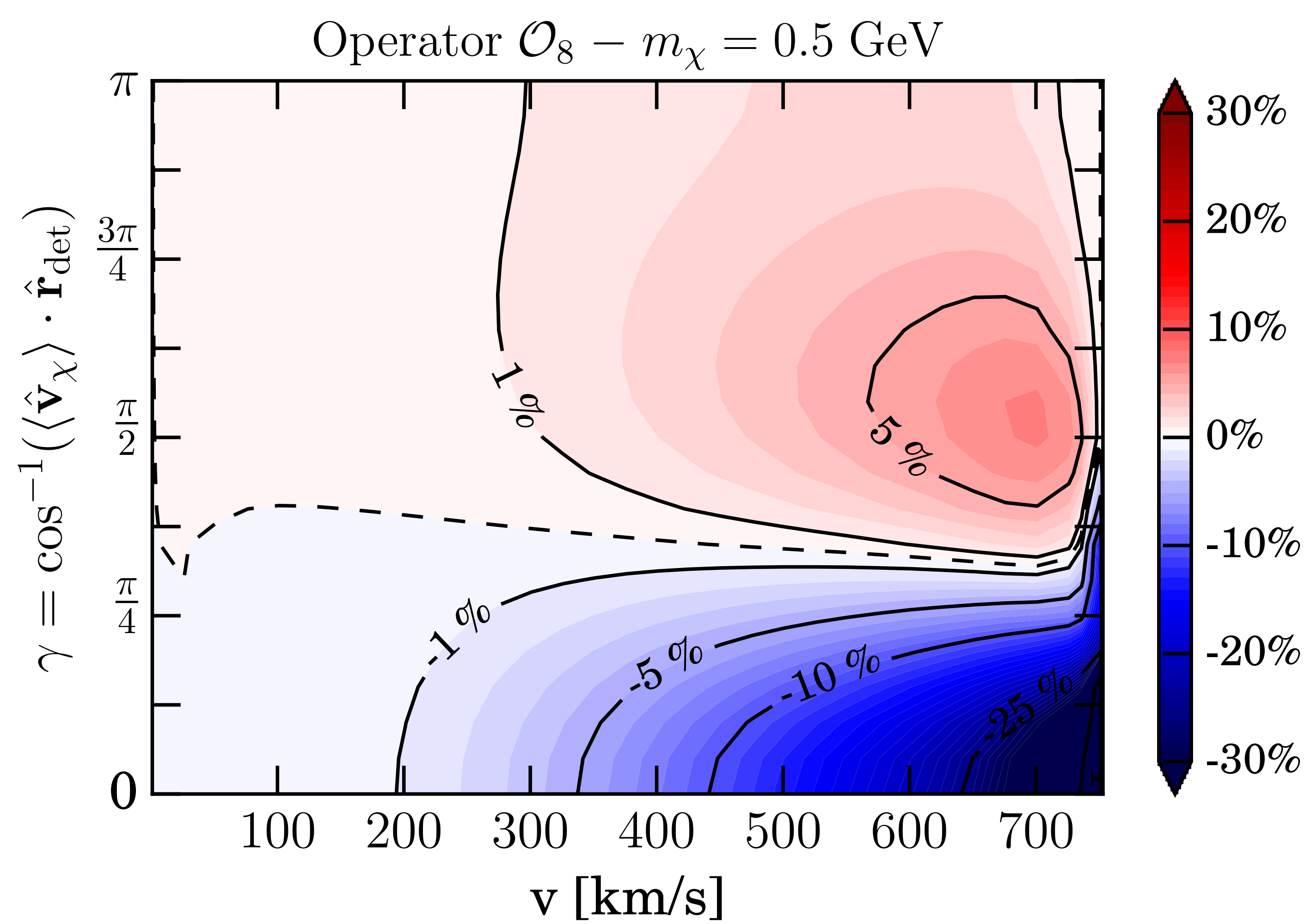


Speed Distribution - O_1 vs O_8

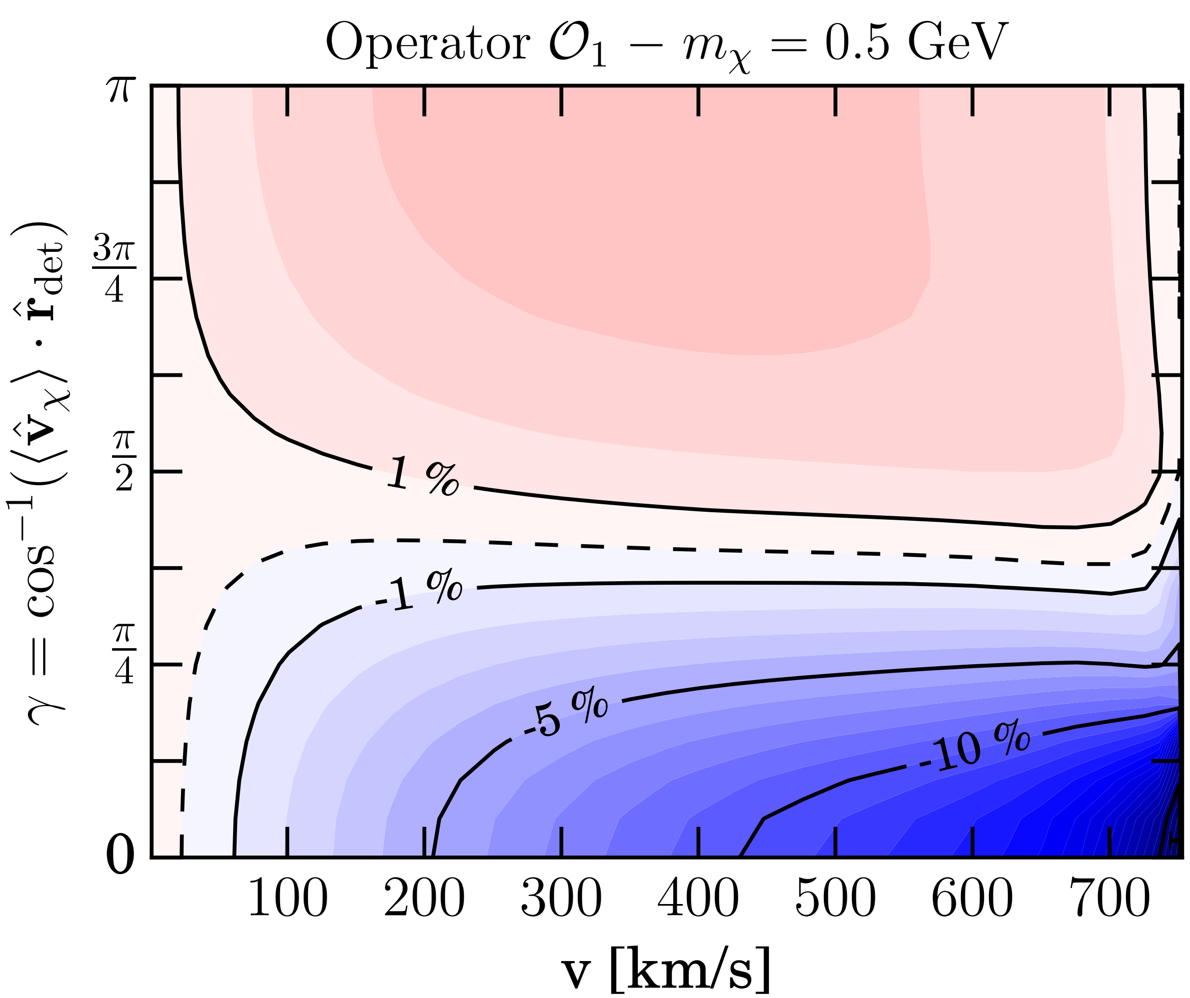


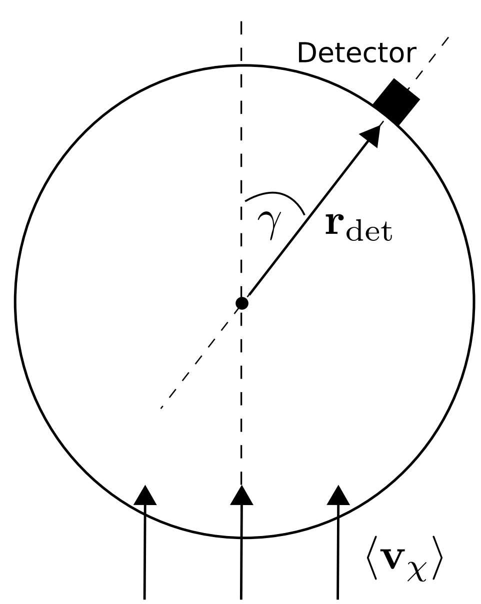


Operator 8 -
preferentially *forward* deflection

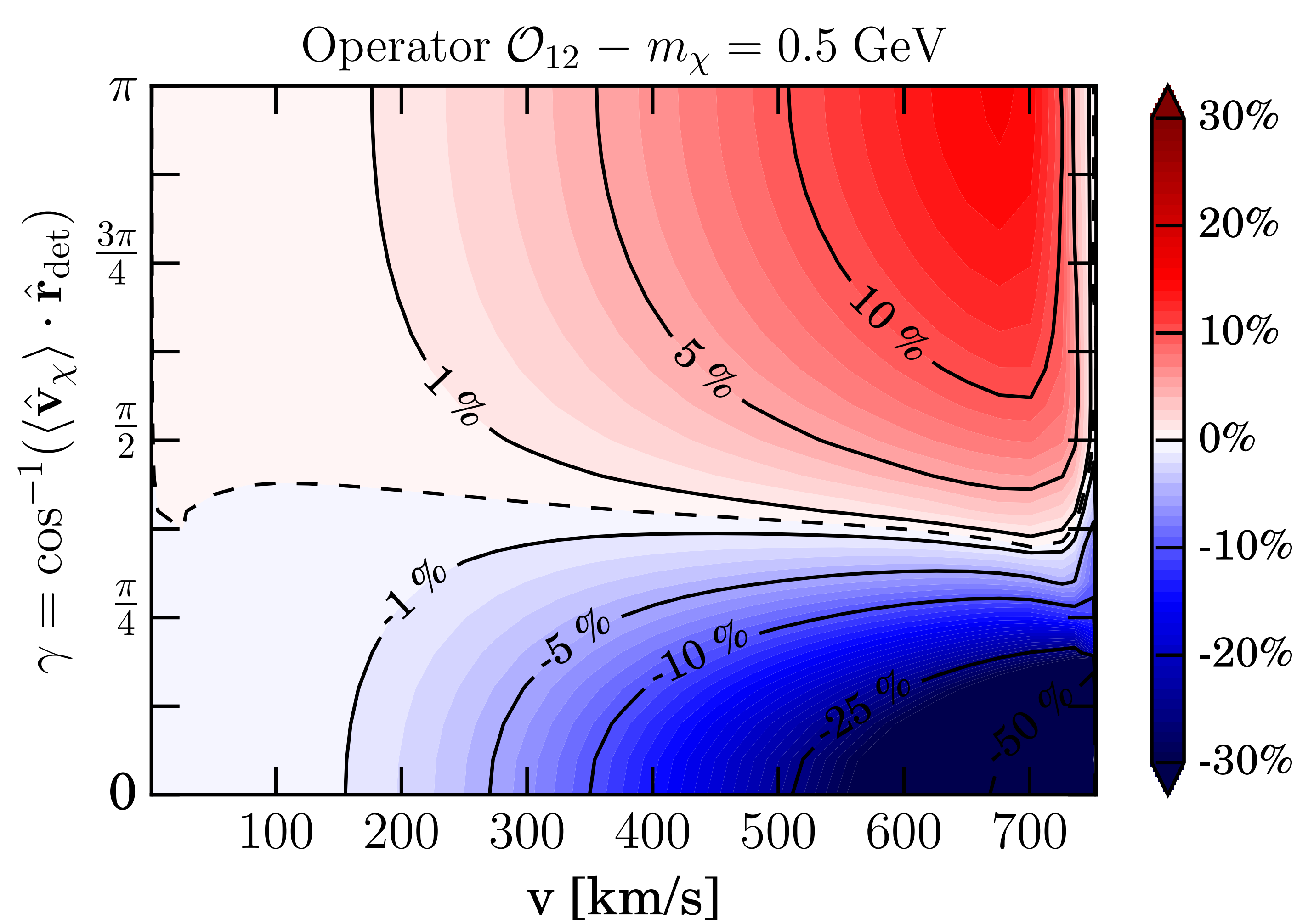


Speed Distribution - O1 vs O12

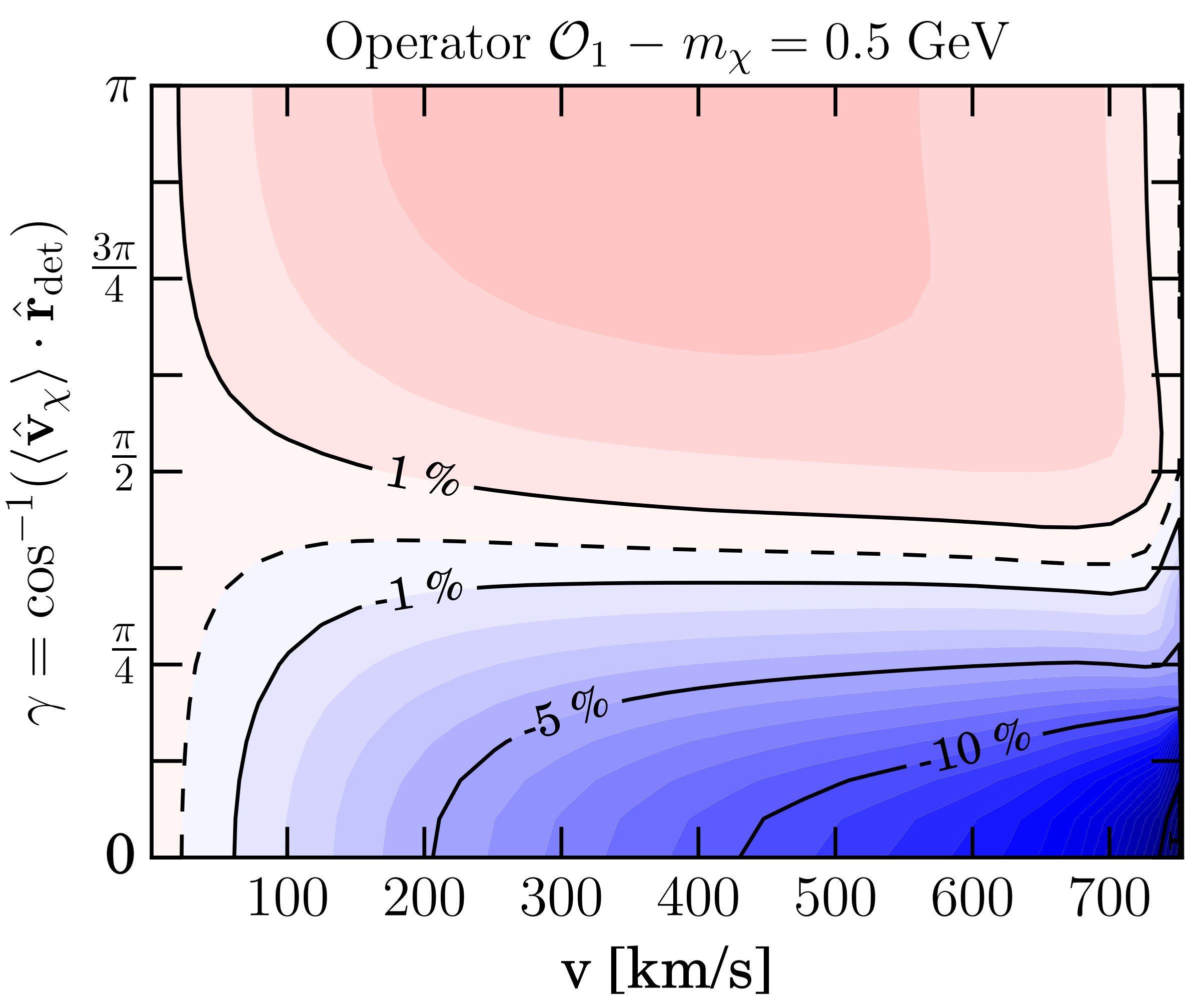


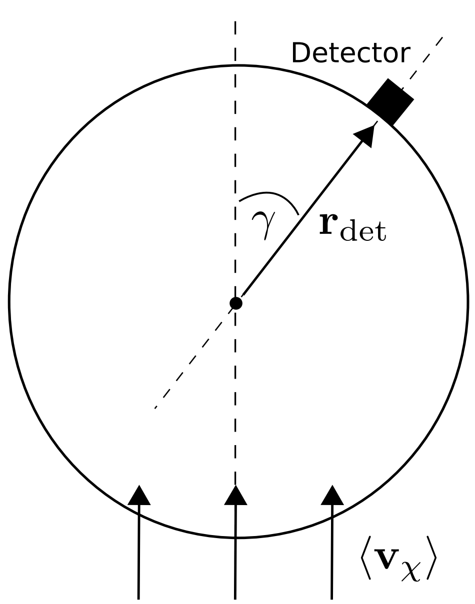


Operator 12 -
preferentially *backward* deflection

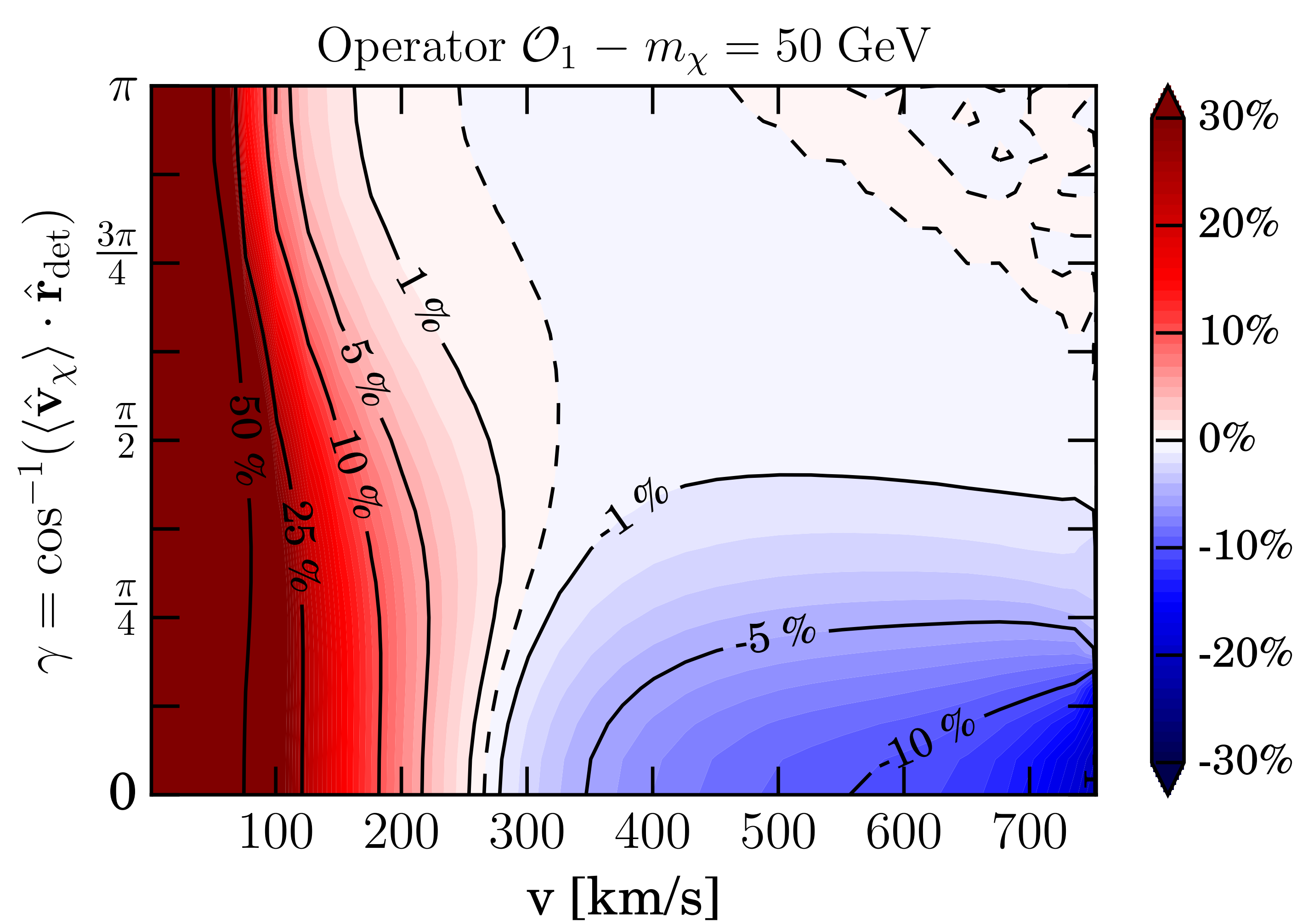


Low mass vs High mass





Higher mass DM



Sanity check

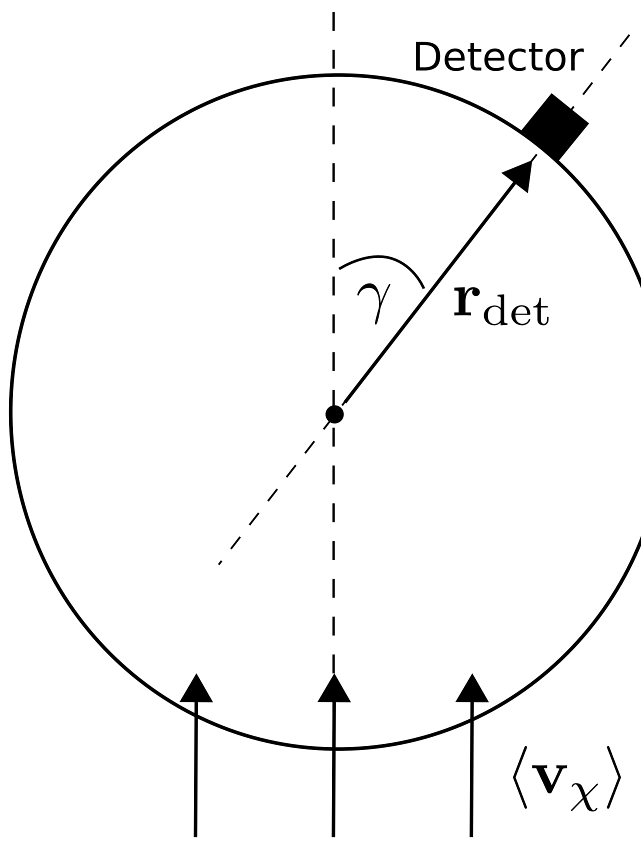
Compare rate of DM particles entering the Earth…

$$\Gamma_{\rm in} = \pi R_\oplus \langle v \rangle$$

…and rate of DM particle leaving the Earth…

$$\Gamma_{\rm out} = \int_{\mathbf{v}\cdot\mathbf{r}>0} \mathrm{d}^2\mathbf{r} \int \mathrm{d}^3\mathbf{v}\, \tilde{f}(\mathbf{v},\mathbf{r})\,(\mathbf{v}\cdot\mathbf{r})$$

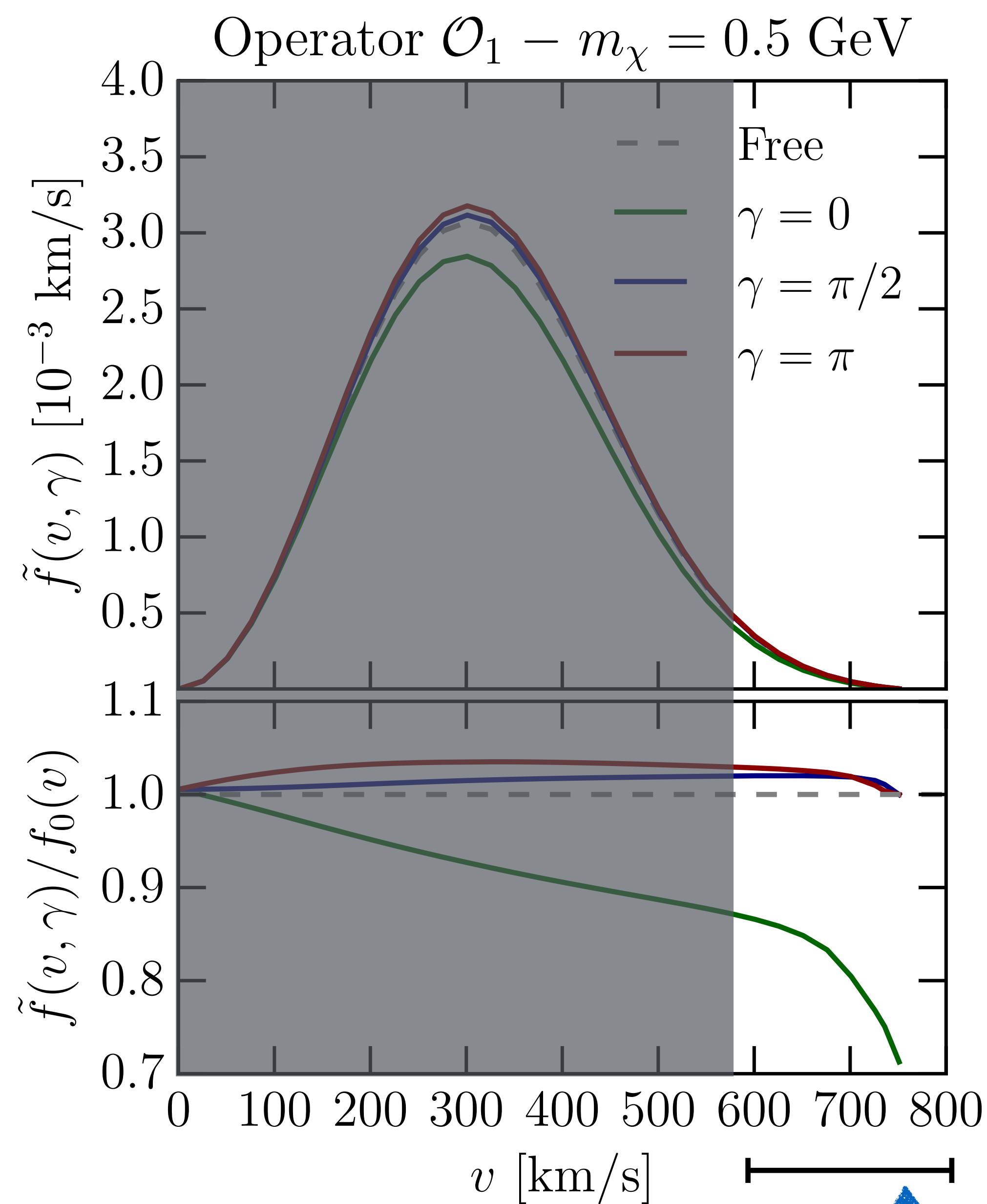
DM mass [GeV]	Operator	$\Delta\Gamma_{\rm out}^{\rm Atten.}/\Gamma_{\rm in}$	$\Delta\Gamma_{\rm out}^{\rm Defl.}/\Gamma_{\rm in}$	$\Gamma_{\rm out}/\Gamma_{\rm in}$
0.5	$\hat{\mathcal{O}}_1$	−7.8%	+7.0%	99.2%
0.5	$\hat{\mathcal{O}}_8$	−8.0%	+7.3%	99.2%
0.5	$\hat{\mathcal{O}}_{12}$	−7.8%	+7.2%	99.4%
50	$\hat{\mathcal{O}}_1$	−7.5%	+7.3%	99.9%
50	$\hat{\mathcal{O}}_8$	−8.0%	+8.4%	100.4%
50	$\hat{\mathcal{O}}_{12}$	−7.3%	+6.6%	99.3%



Event Rate

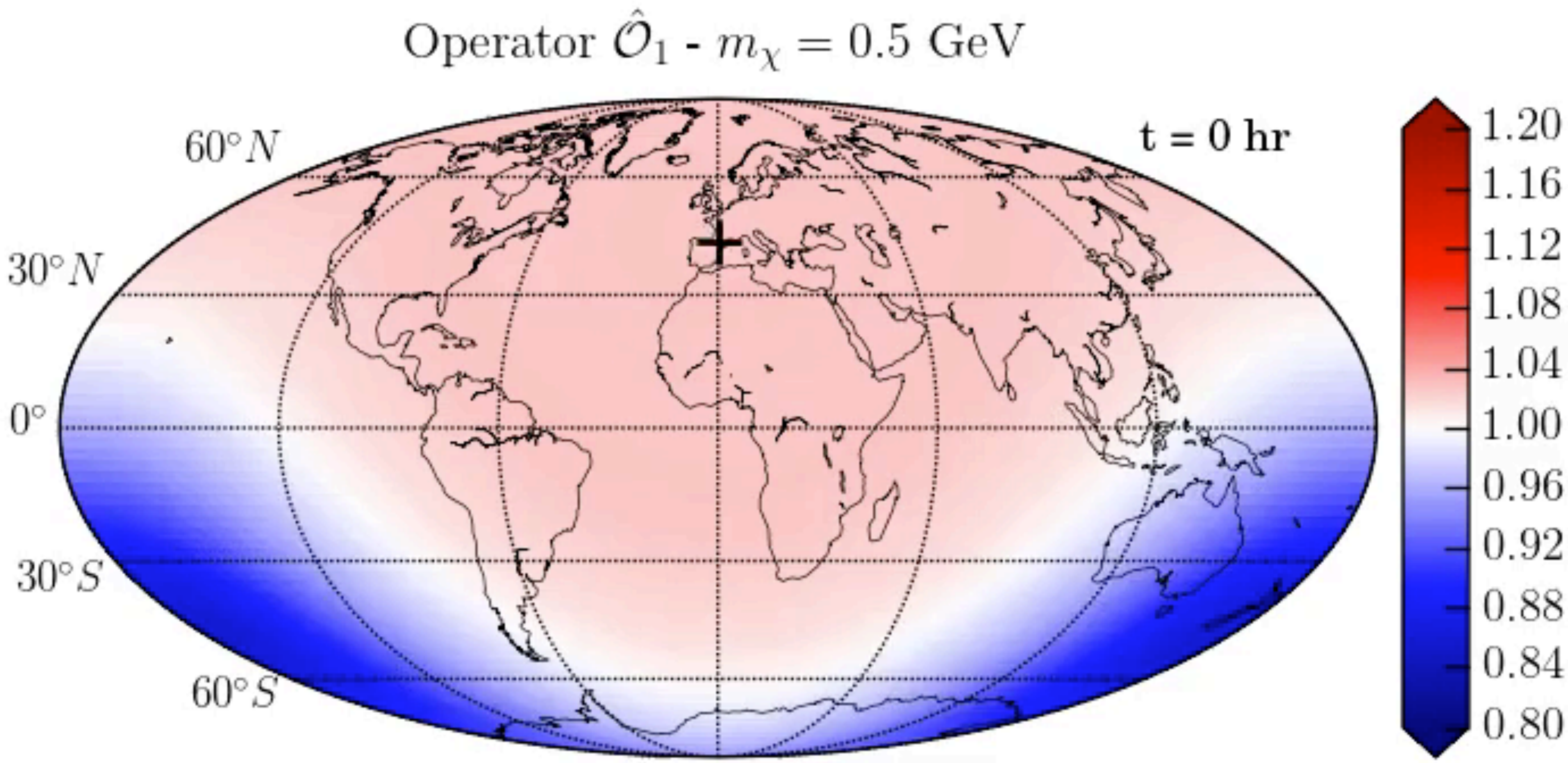
Calculate number of signal events in a CRESST-II like experiment, with and without the effects of Earth-Scattering, N_{pert} and N_{free}.

Scattering predominantly with Oxygen and Calcium.



DM particles within $3\,\sigma_E$ of the energy threshold
$E_{\mathrm{th}} \sim 300$ eV

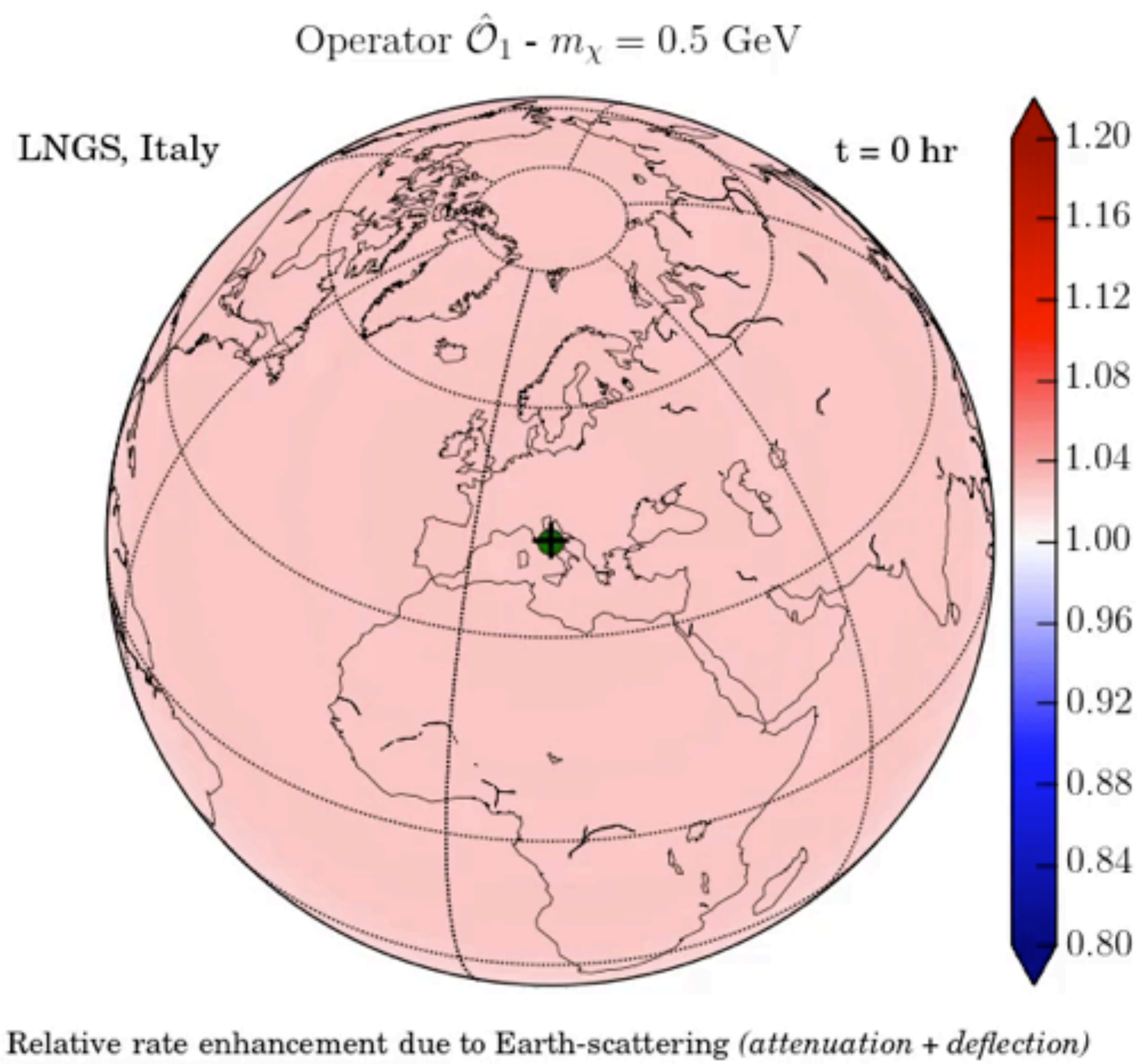
Mapping the CRESST-II Rate



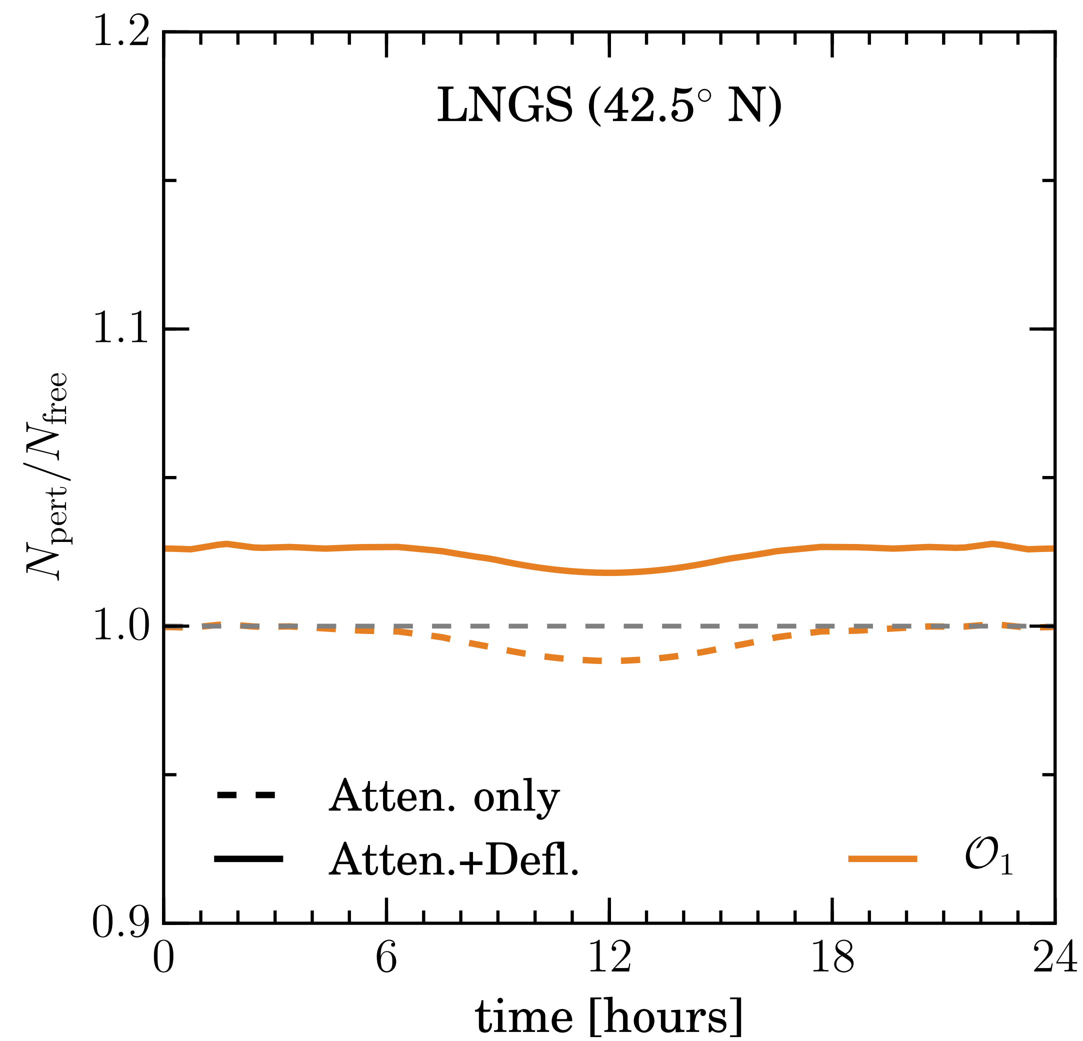
Relative rate enhancement due to Earth-scattering ***(attenuation + deflection)***

LNGS - Operator 1

LNGS - Gran Sasso Lab, Italy

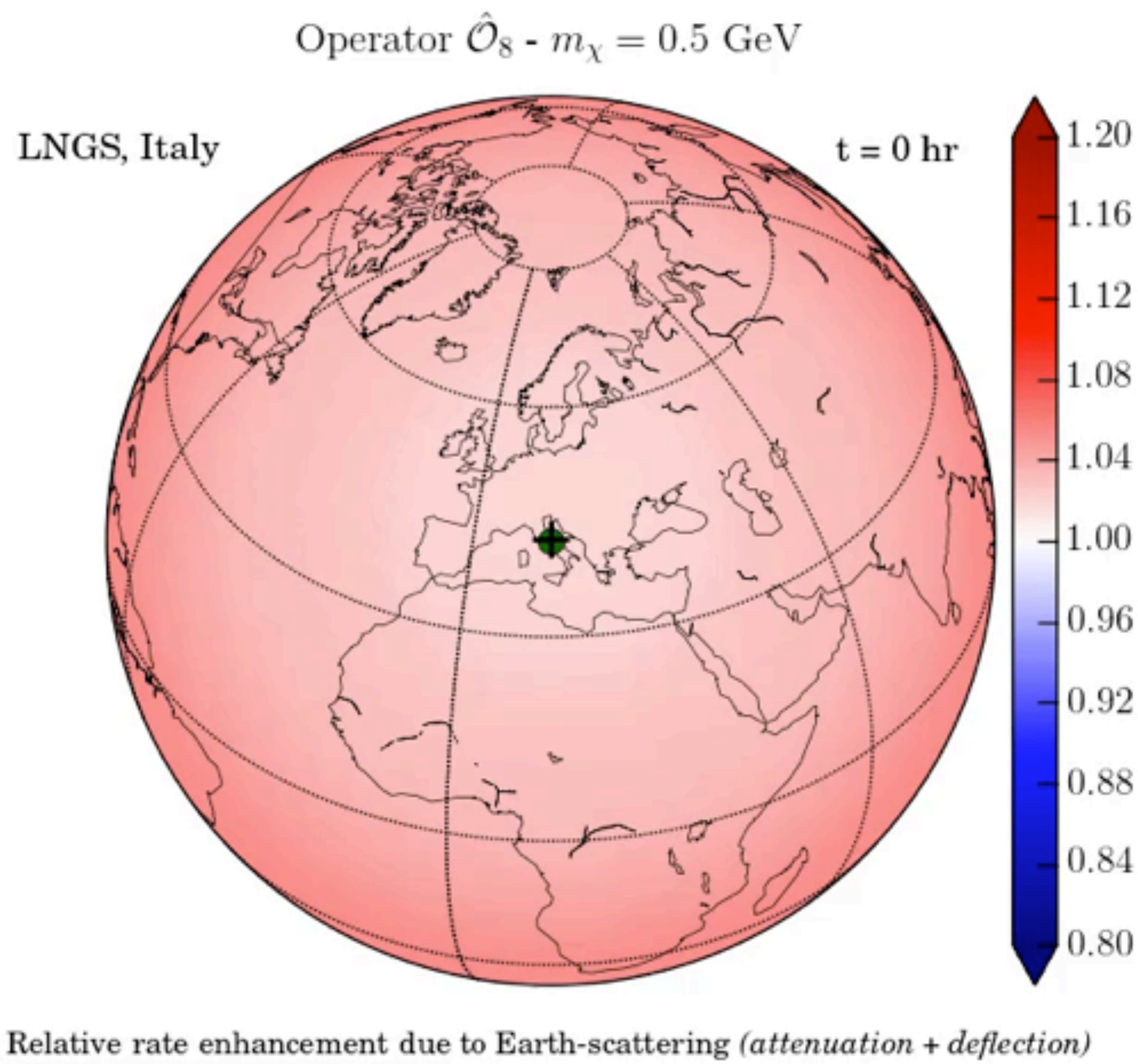


Operator 1 - isotropic deflection

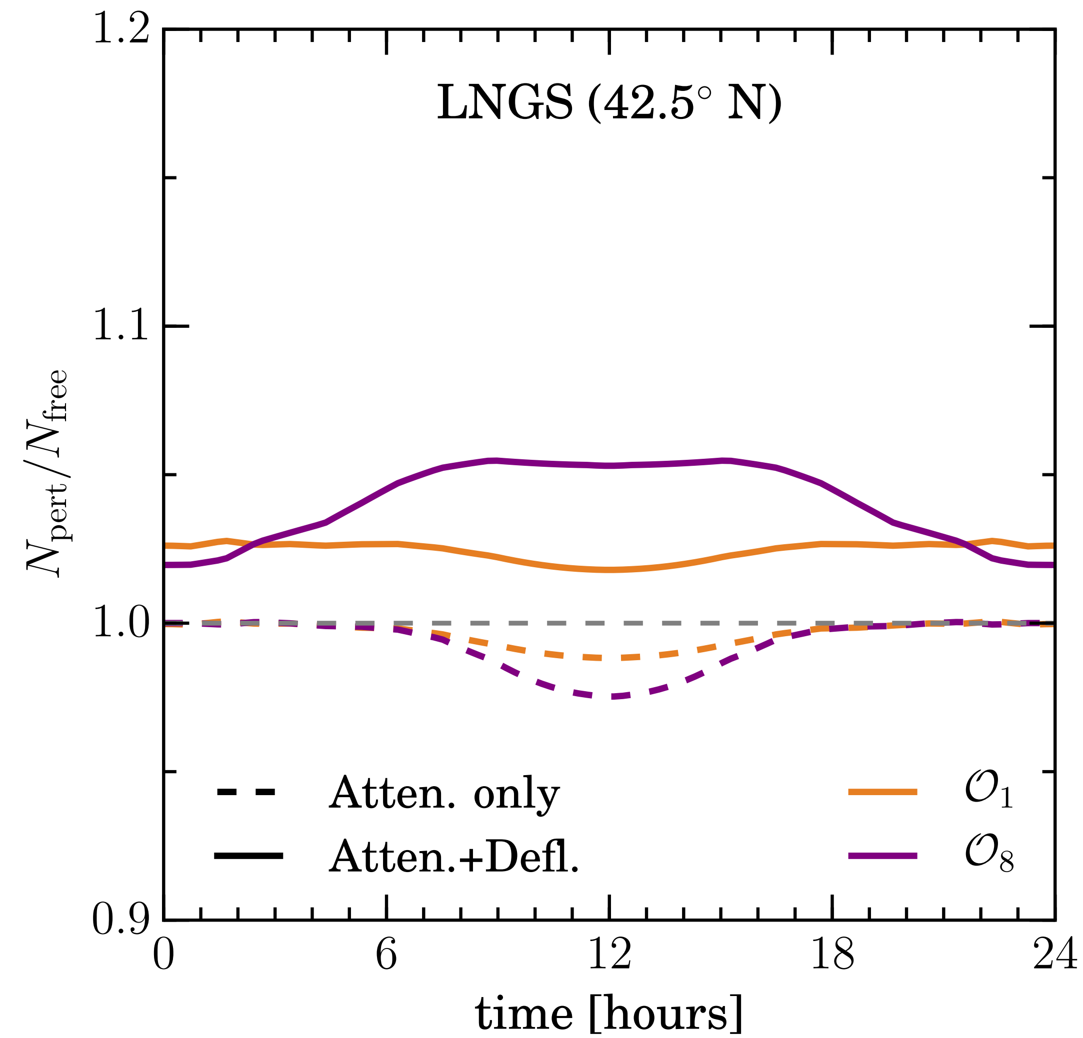


LNGS - Operator 8

LNGS - Gran Sasso Lab, Italy

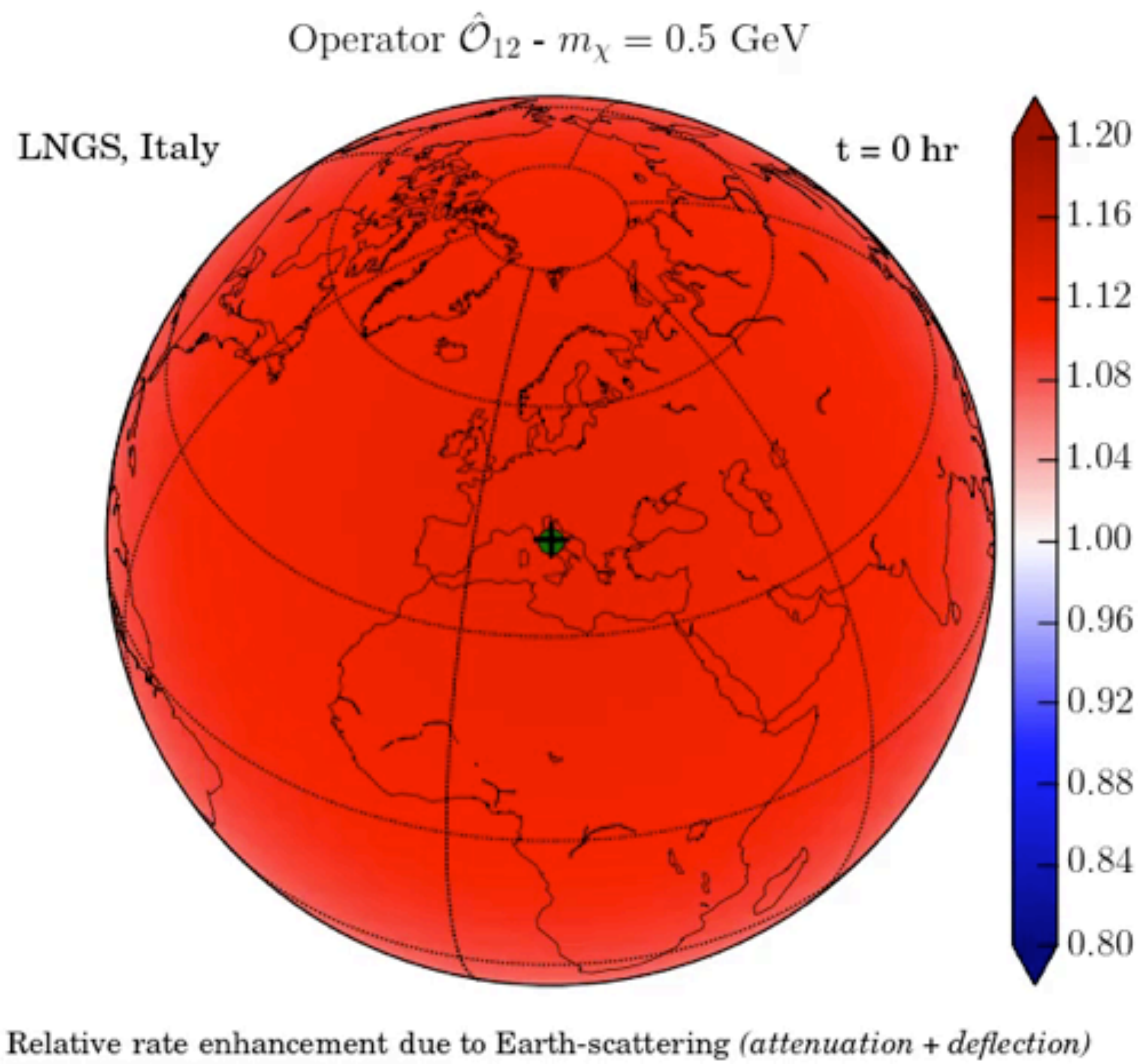


Operator 8 - forward deflection

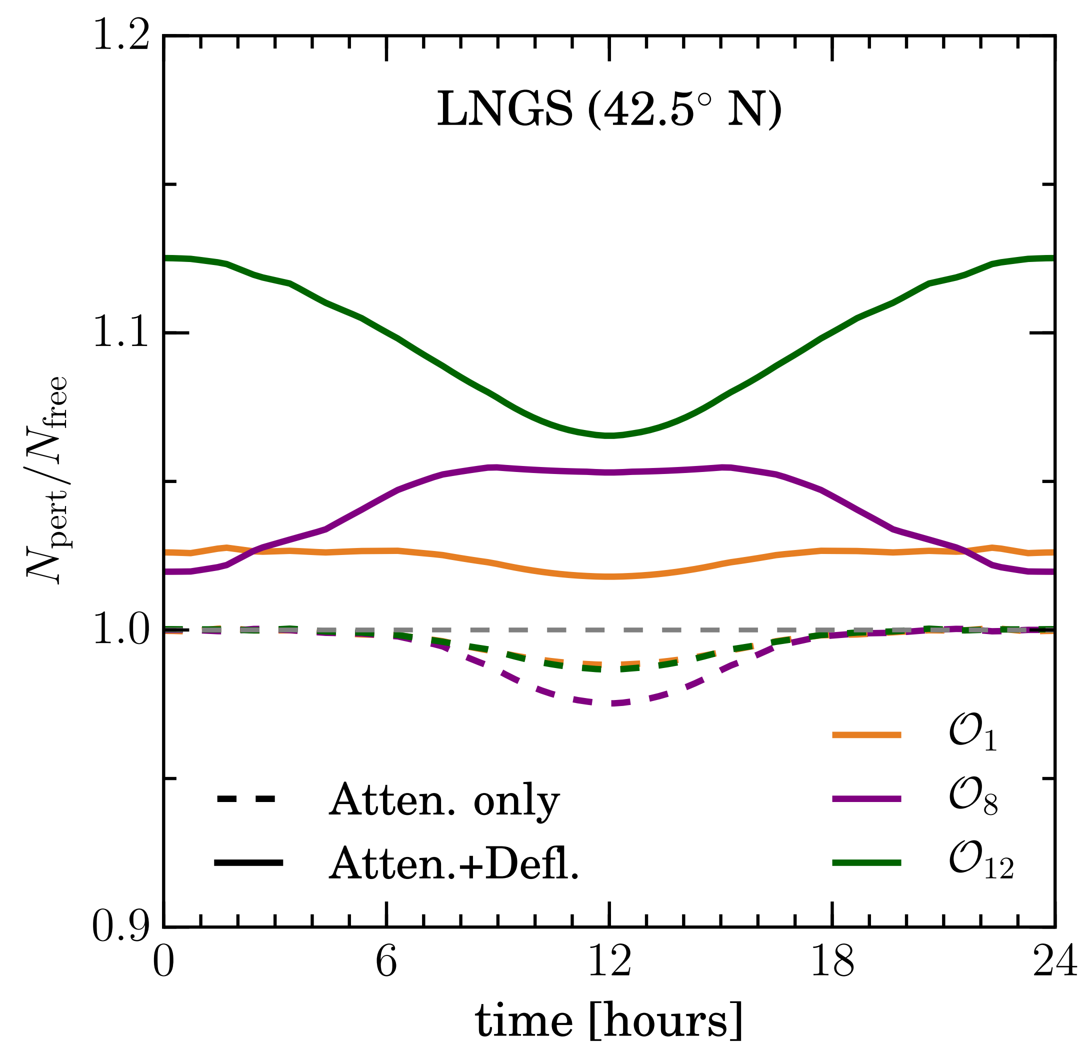


LNGS - Operator 12

LNGS - Gran Sasso Lab, Italy

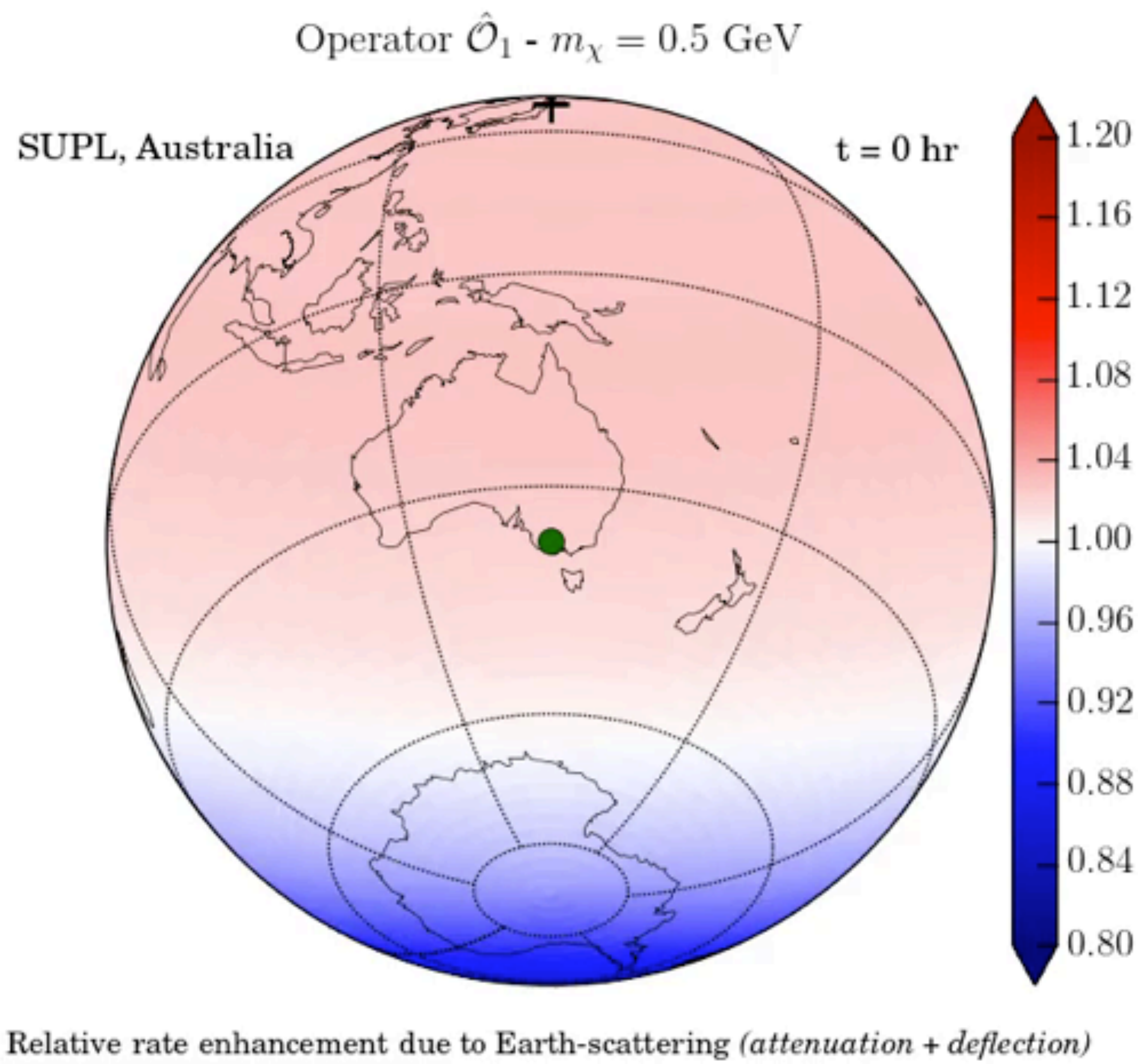


Operator 12 - backward deflection

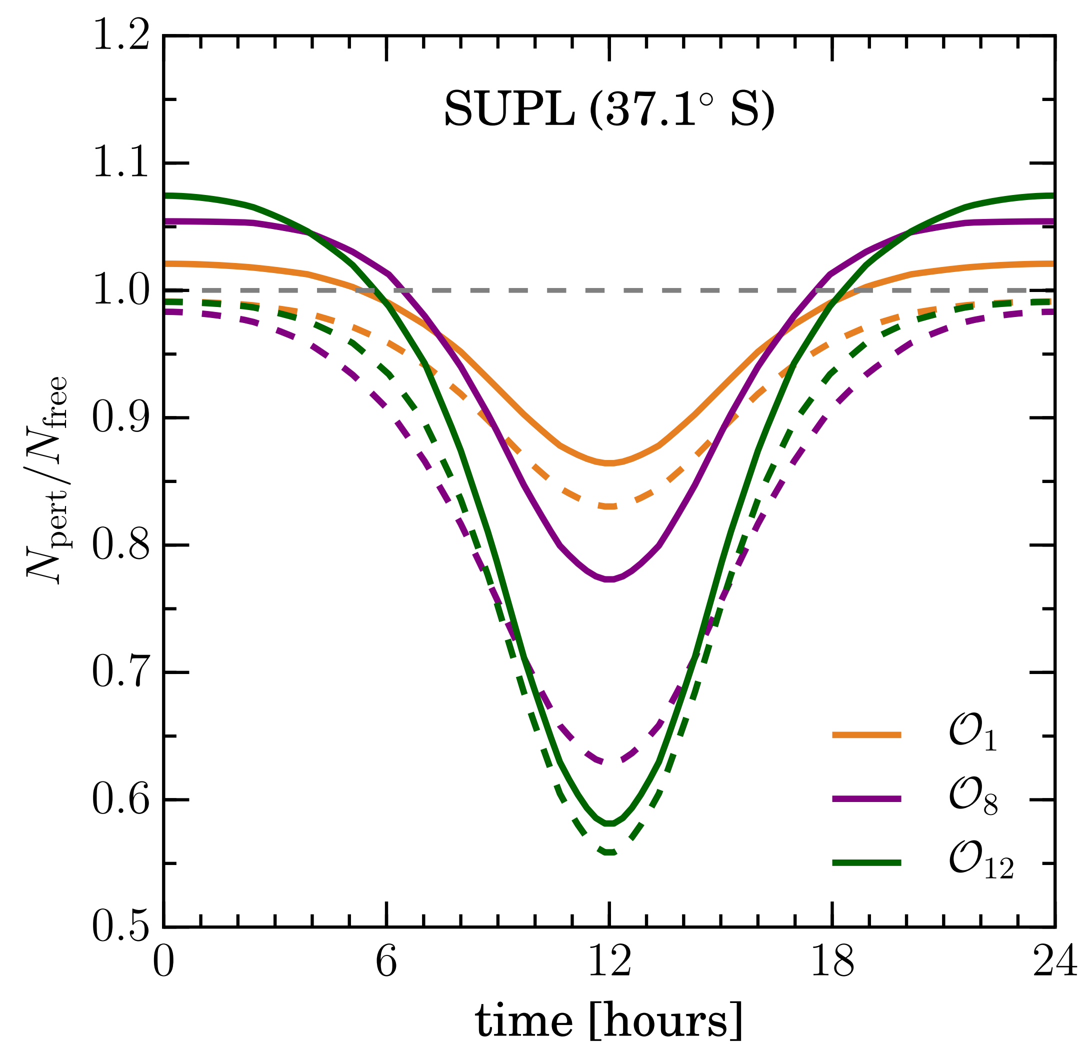


SUPL - Operator 1

SUPL - Stawell Underground Physics Lab, Australia

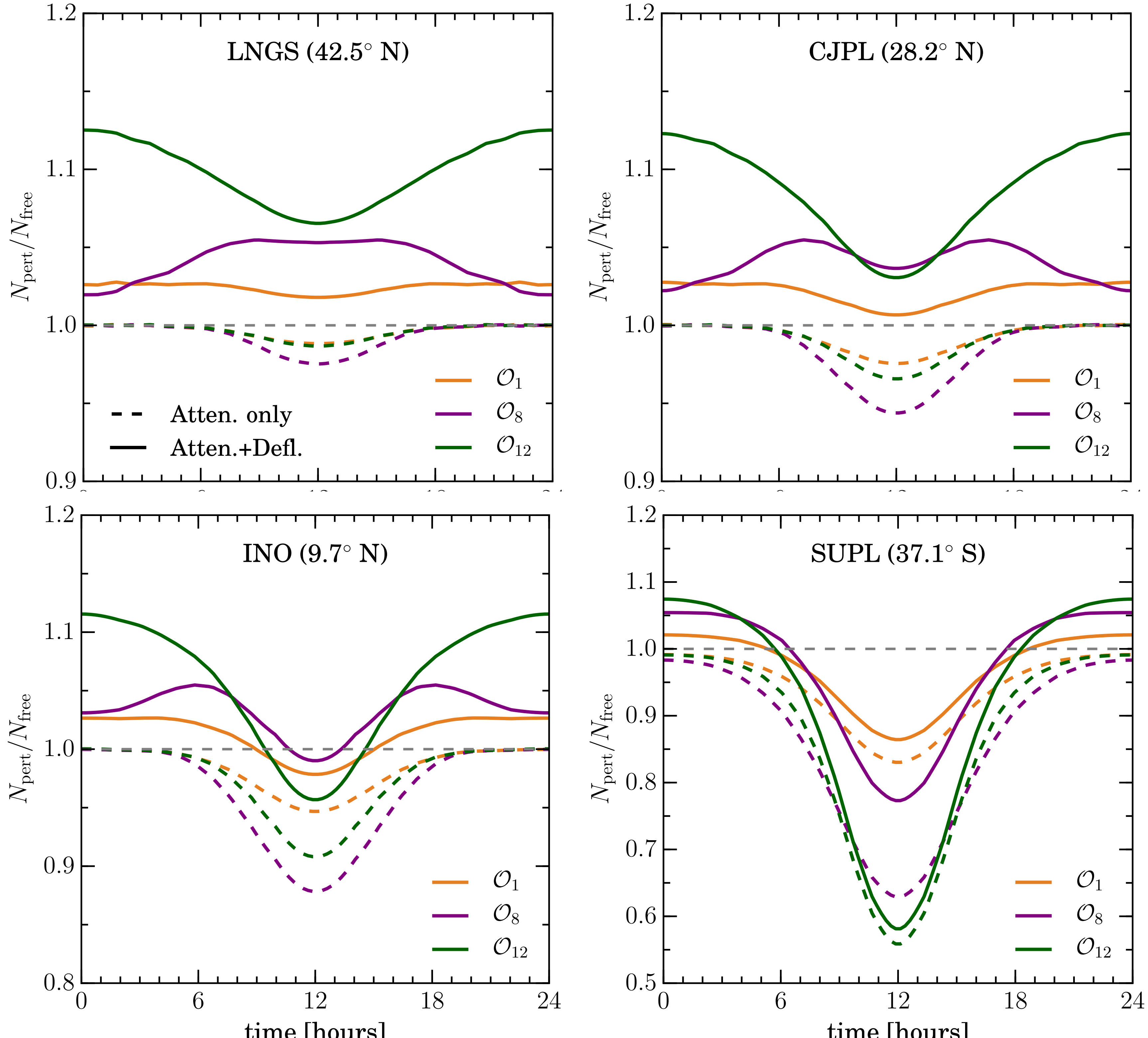


Operator 1 - isotropic deflection



Around the world

China Jinping Lab

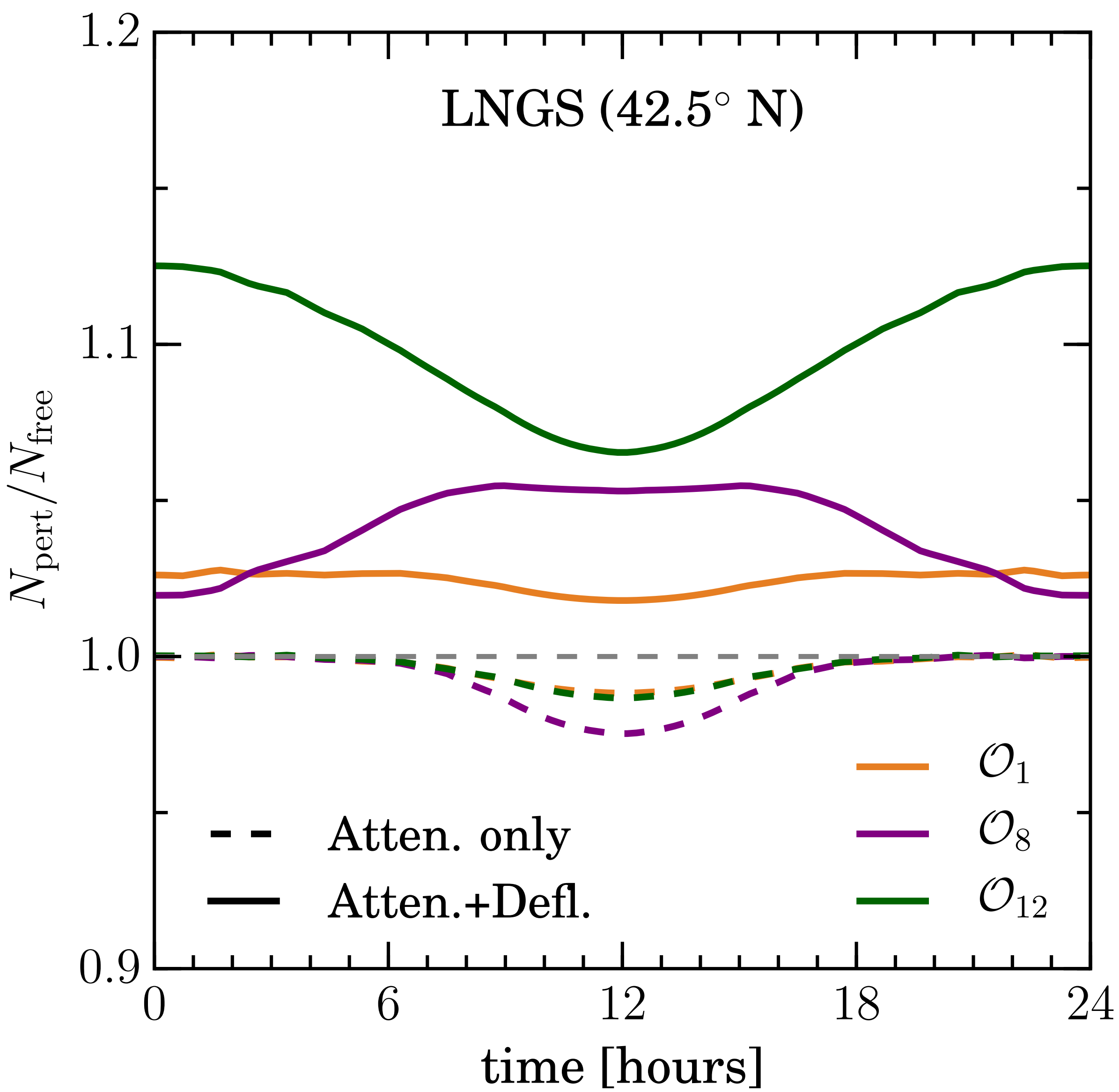


India-based
Neutrino Observatory

Implications of Earth-Scattering

Careful calculation (including deflection and attenuation)
in the 'single-scatter' regime

BJK, Catena & Kouvaris
[1611.05453]



Smoking gun signature:
daily modulation
+ location dependence
could confirm DM nature

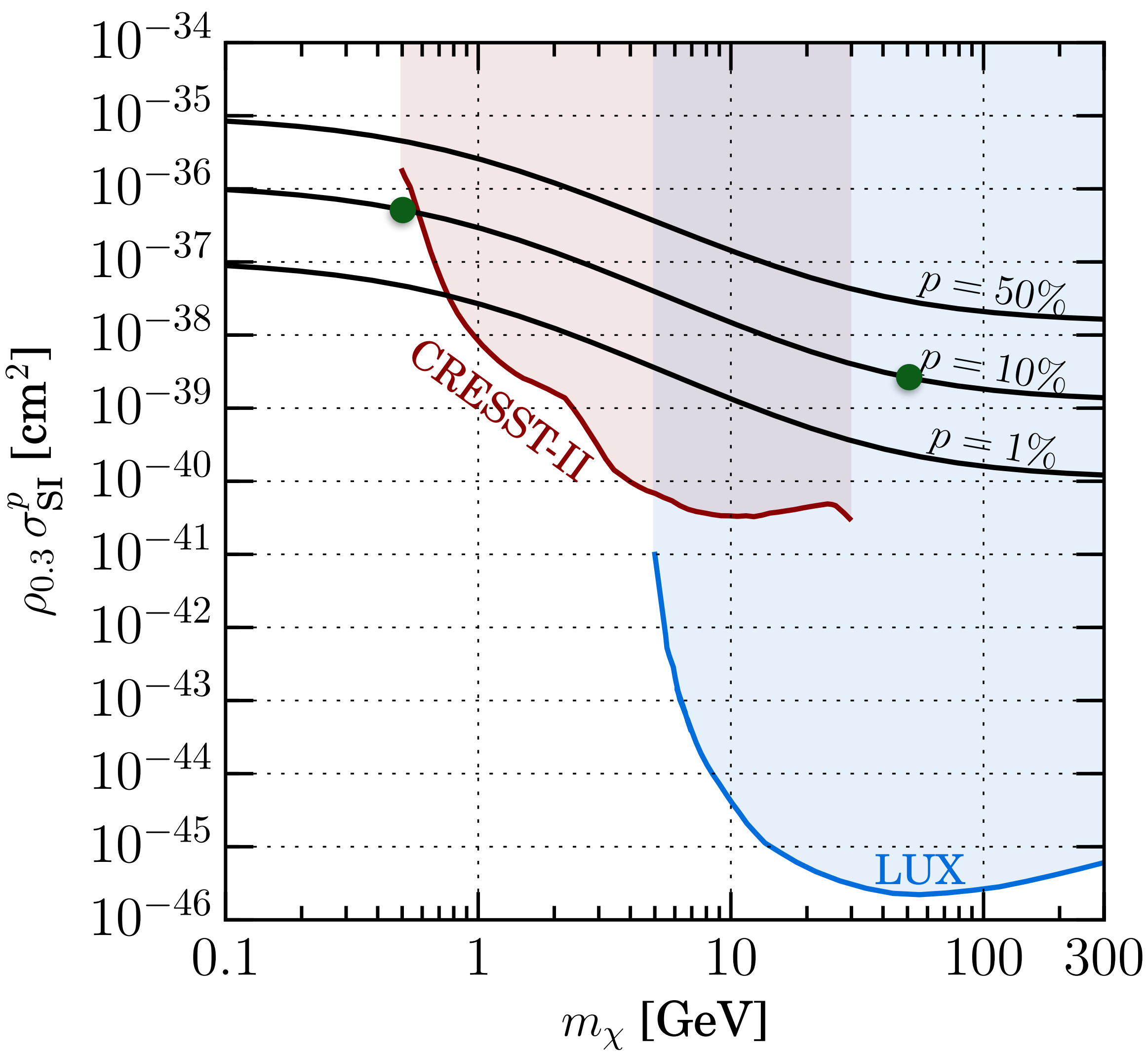
Possibility to distinguish different interactions with different amplitude and phase of modulation

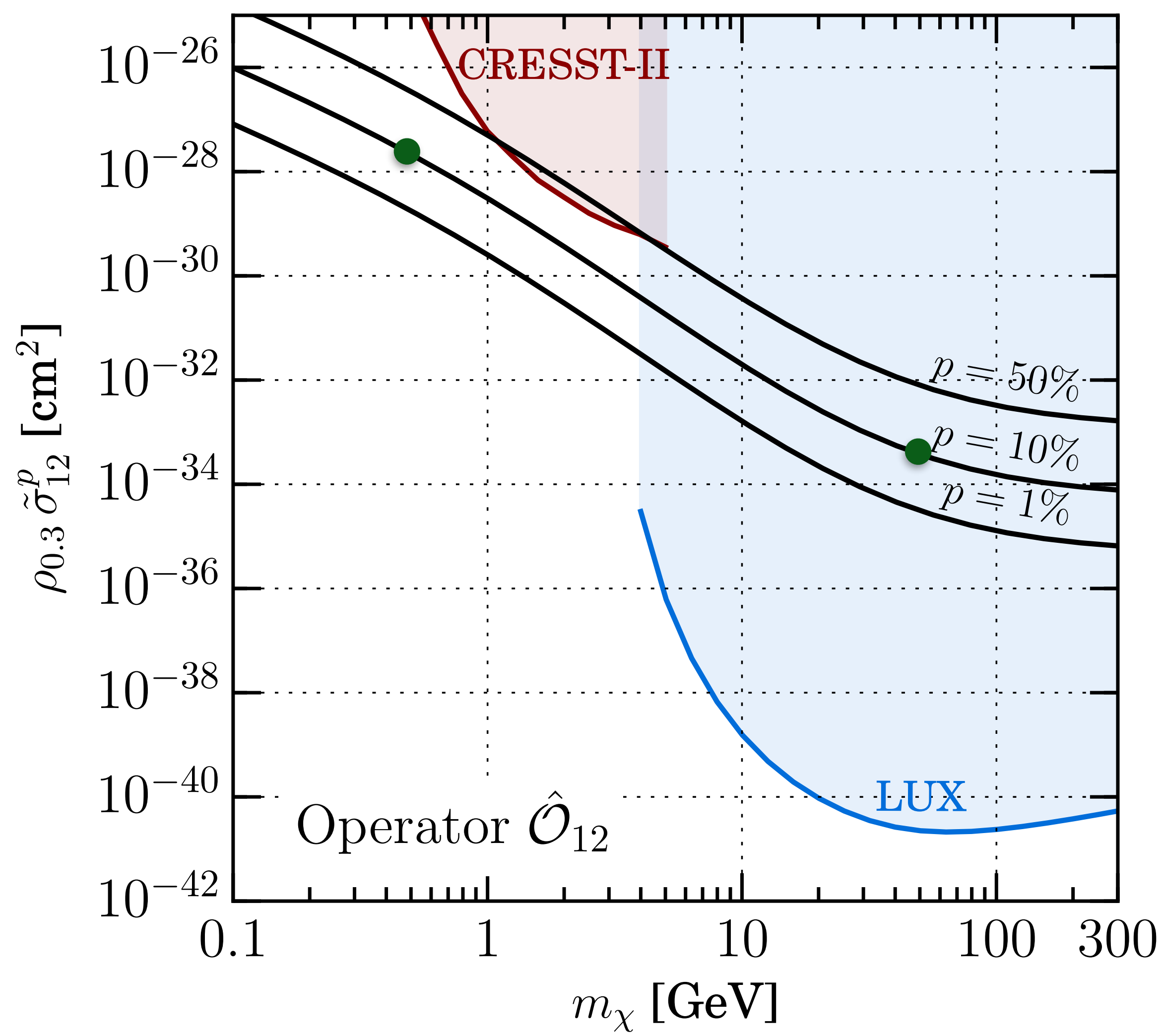
EARTHSHADOW code available online to include these effects:
github.com/bradkav/EarthShadow

Ideas for the future

Mapping out the parameter space

Continue mapping out parameter space (m_χ, σ_p) and explore impact on upper limits for a range of interactions…





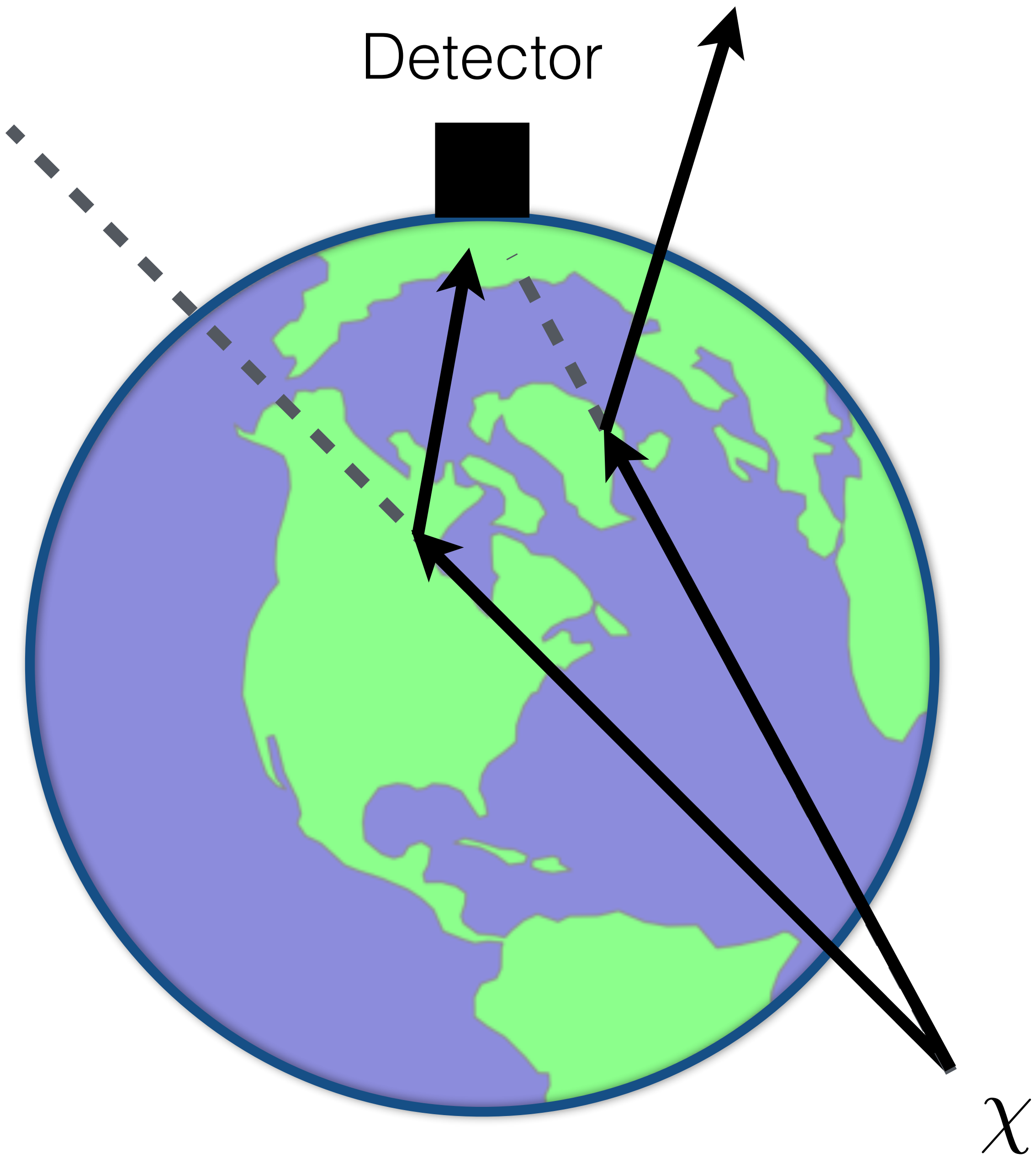
…and encourage experimental collaborations to explore full NREFT parameter space.

Directionality

Distortion of $f(\mathbf{v})$ should also lead to a directional signature

Studied previously for very efficiency stopping

[1509.08720]



Recent proposal for directional sensitivity to low mass DM using semiconductor detectors

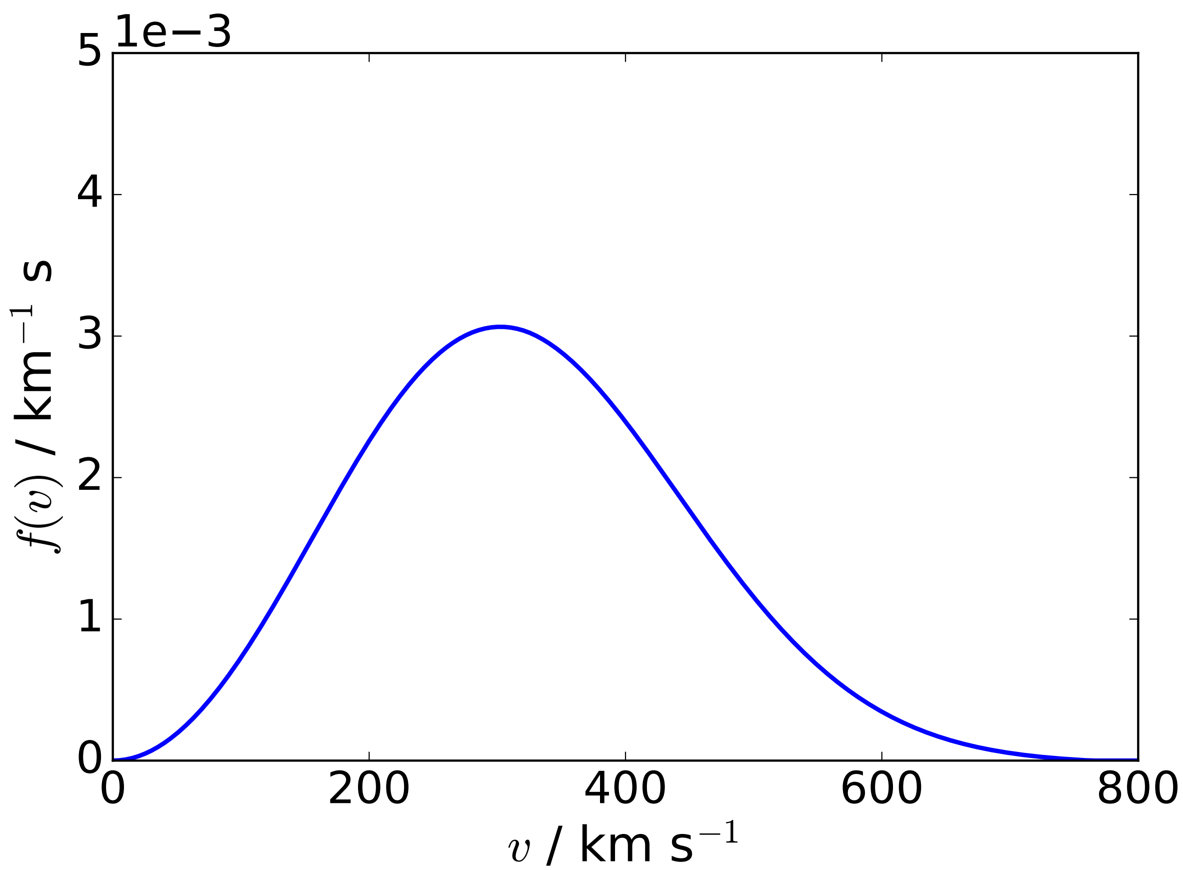
[1703.05371]

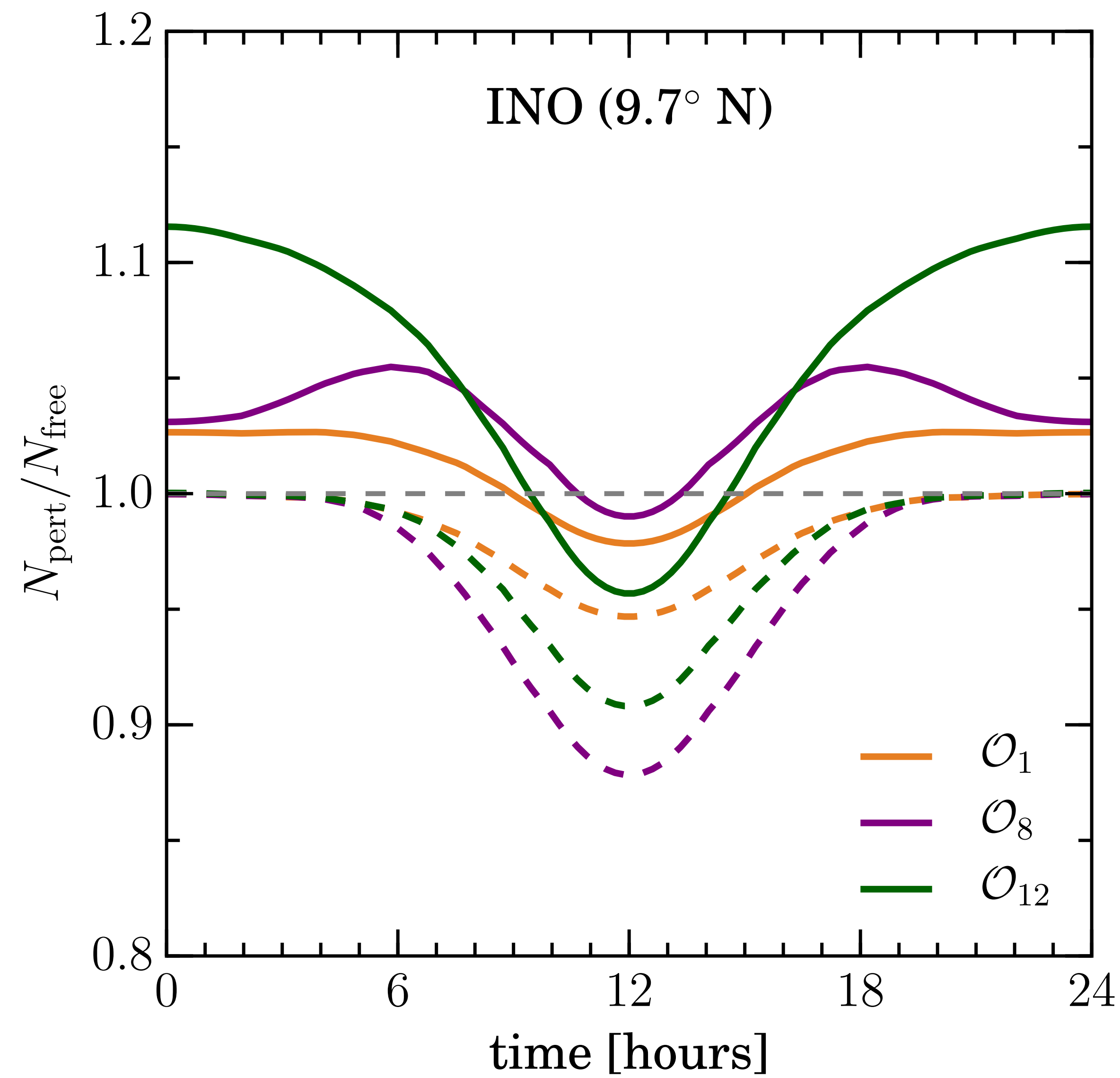
In our case, Earth-Scattering should give an excess of particles originating from the 'downward' direction (depending on time of day)

Astrophysical Uncertainties

How robust are these results against changes
to the (free) velocity distribution?

Doesn't depend on spectral information, only timing information.

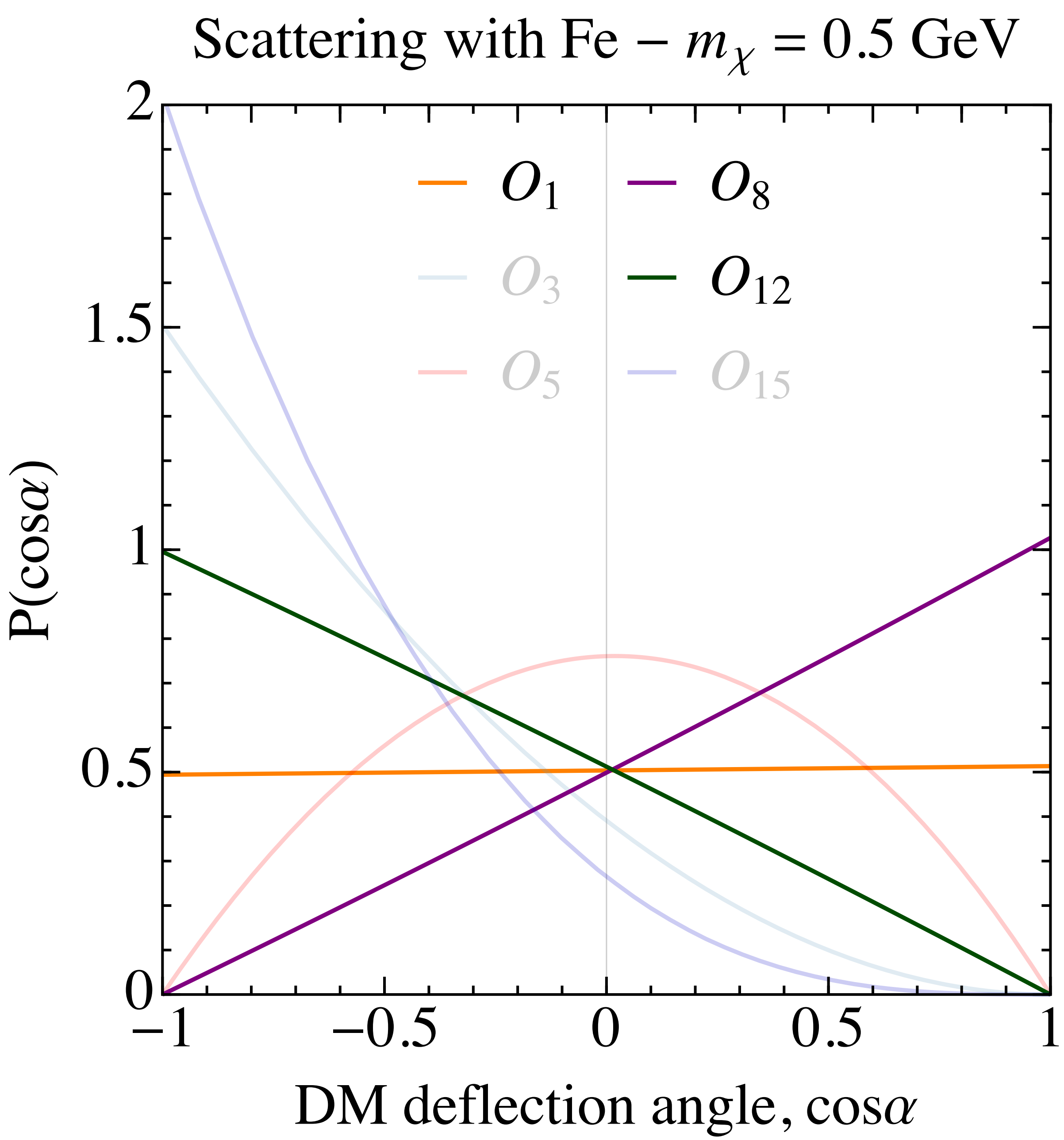




Also, what about degeneracy between cross section and DM density…?

Low mass Dark Matter

The 'many-scatter' regime for low mass DM?



Backward

Forward

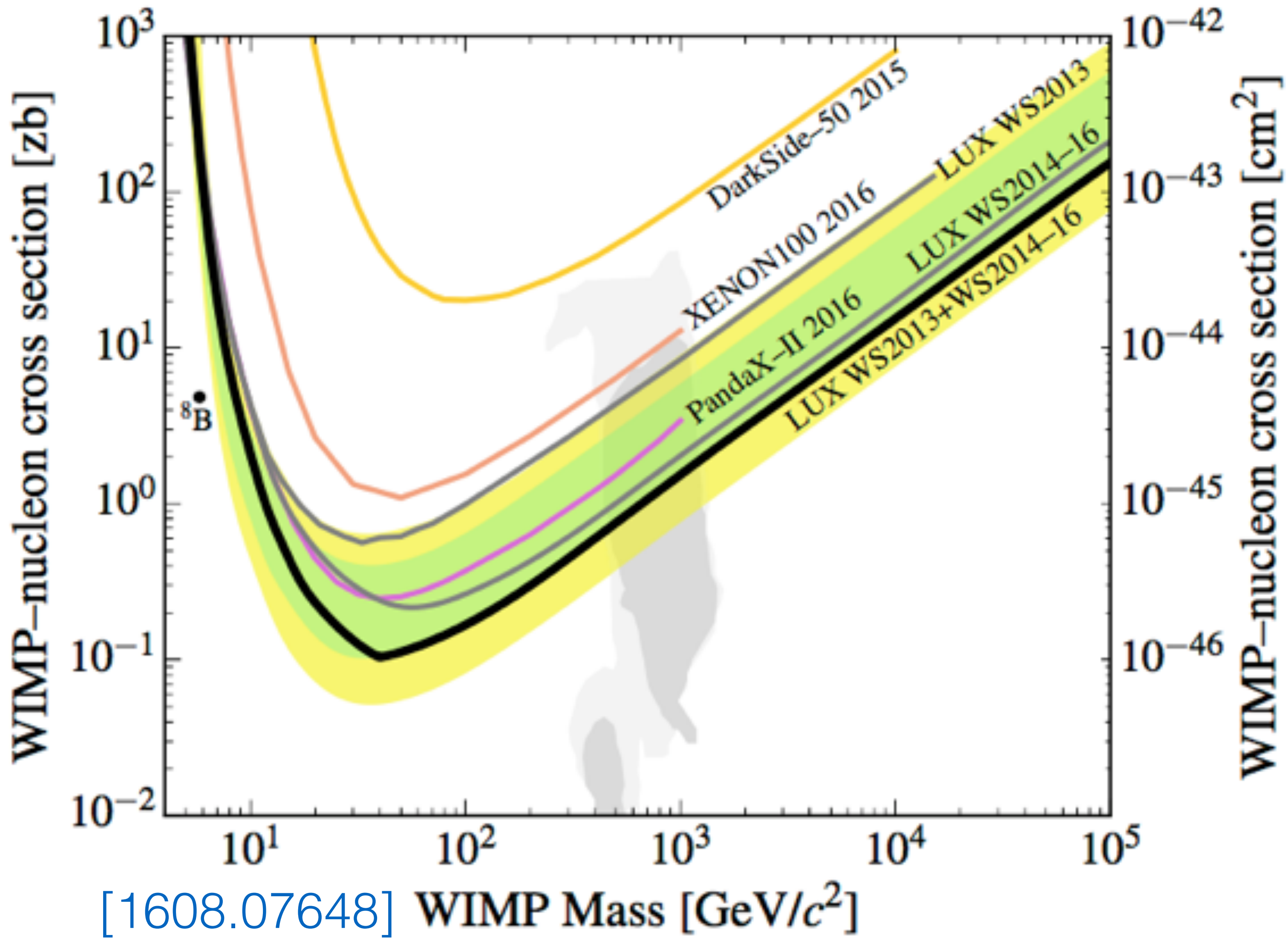
Low mass DM loses almost no energy on scattering

For standard SI interactions the scattering is isotropic

Should be able to model as a random walk/diffusion process

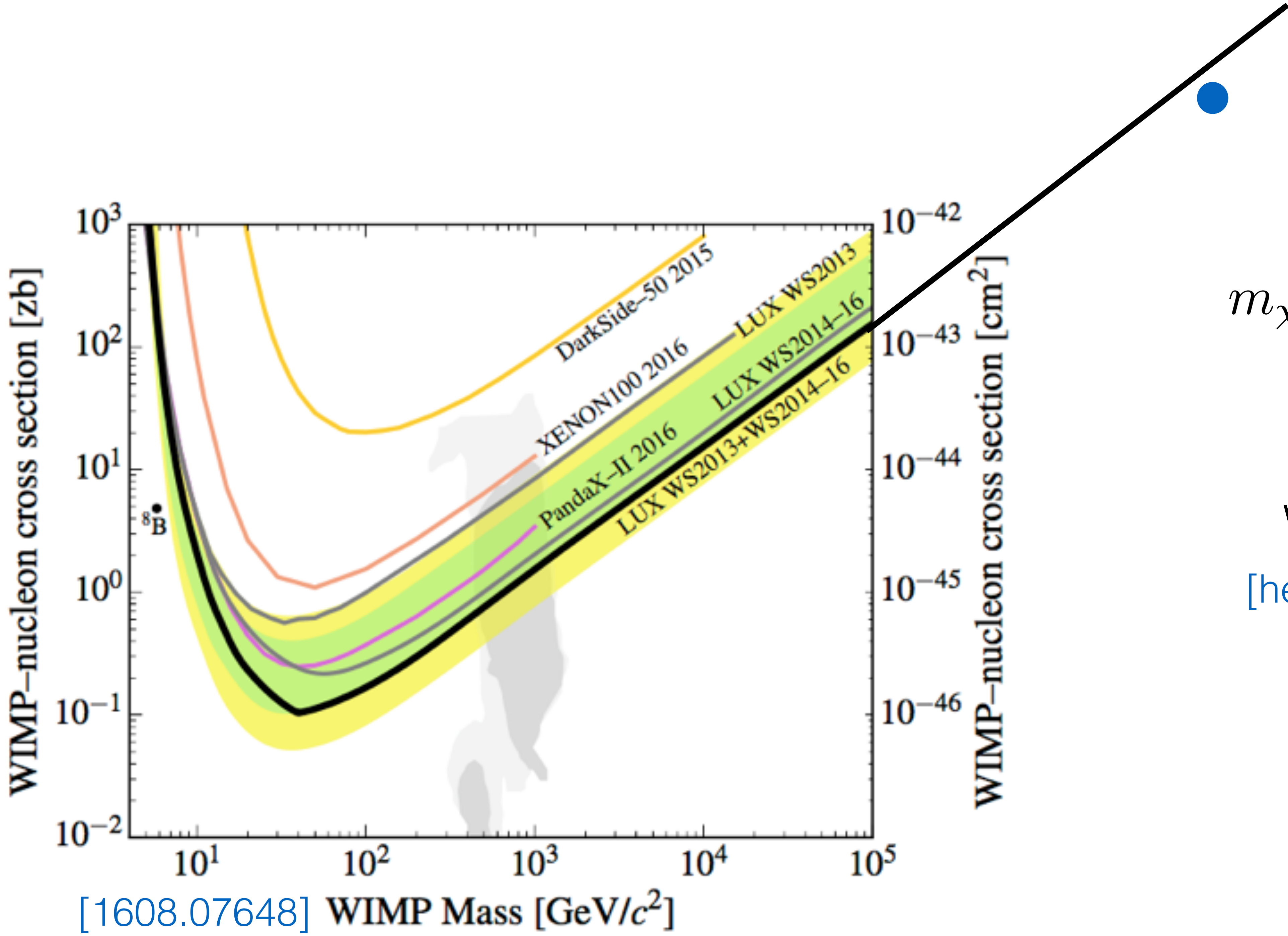
High mass Dark Matter

What about very heavy DM?



[1608.07648]

What about very heavy DM?



[1608.07648]

$p_{\rm scat} \sim 1$

for

$m_\chi \sim 10^7$ GeV

WIMPzillas!

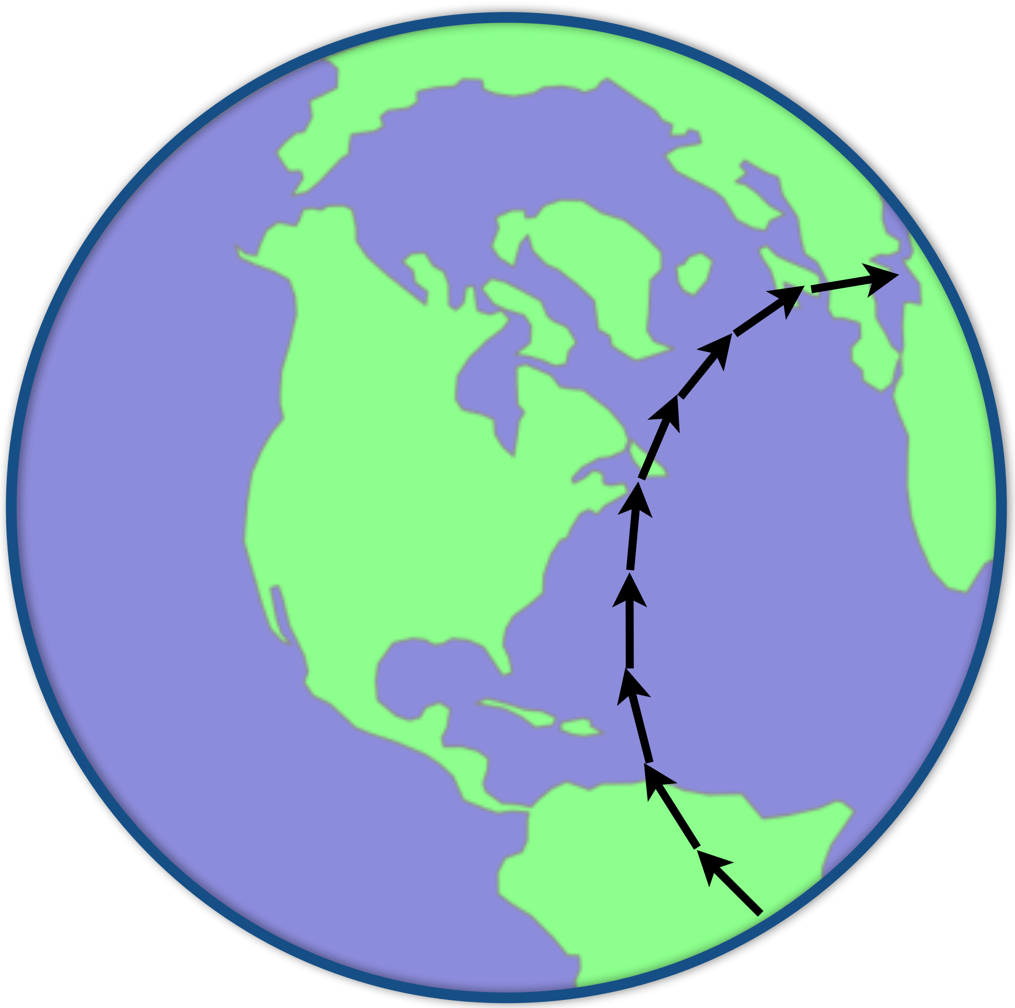
[hep-ph/9810361, 1606.00923]

High mass Dark Matter

What about very heavy DM?

In the limit, $m_\chi \to \infty$ DM is *not deflected* and loses *no energy* when scattering with Earth nuclei

But for finite m_χ, get a small deflection and energy loss.

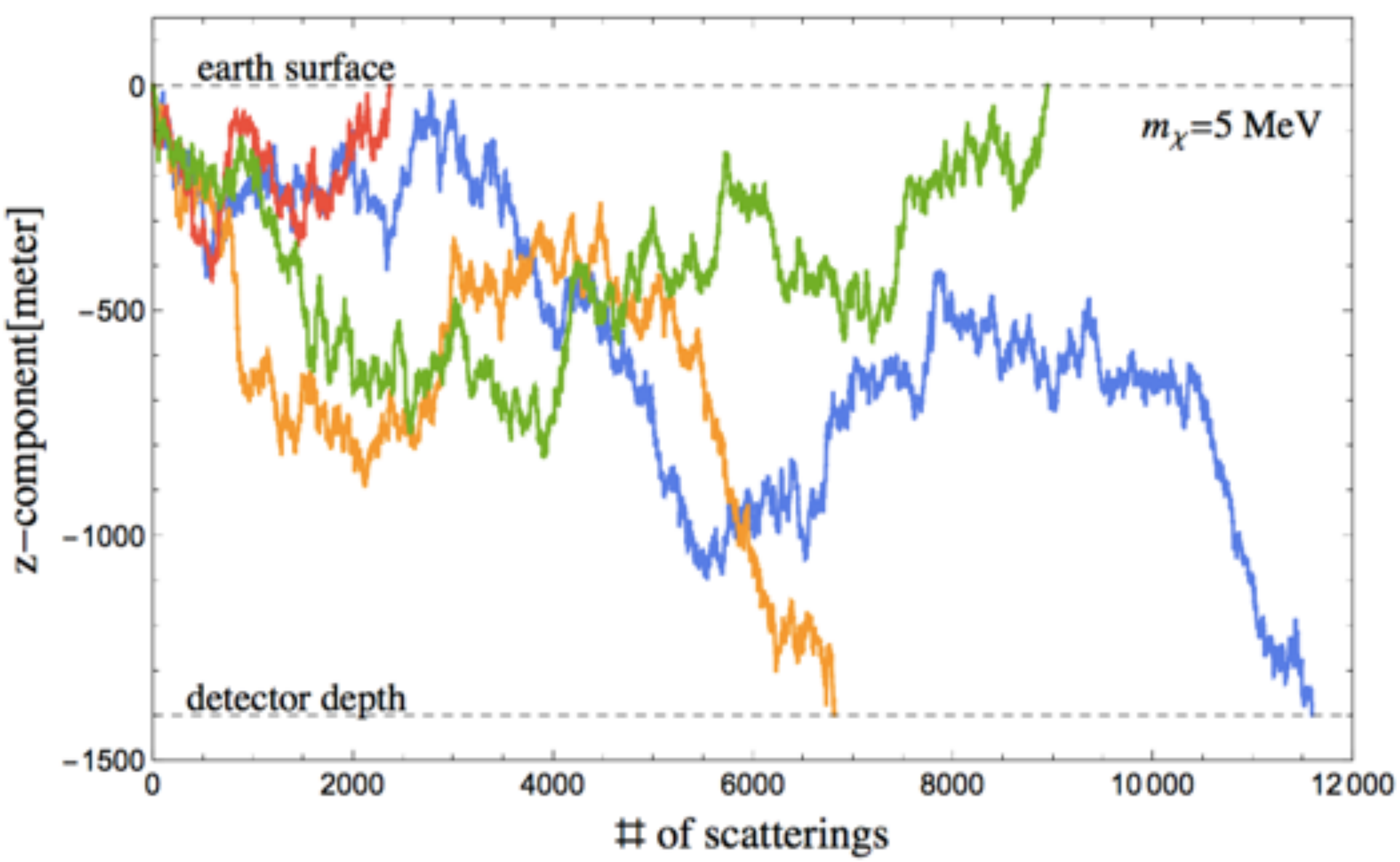


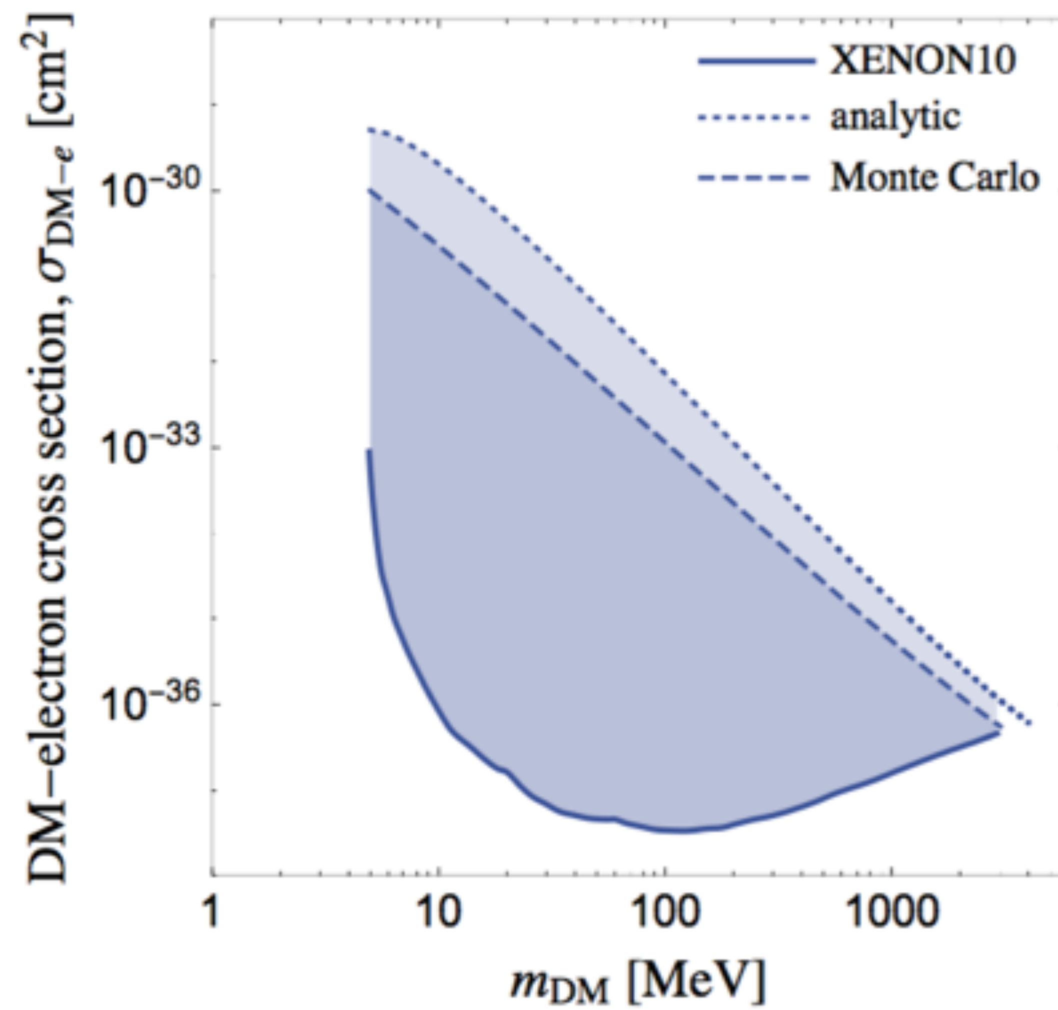
Heavy DM effectively follows smooth, curved trajectories through the Earth

Monte Carlo Simulations

State-of-the-art MC simulations are currently in development -
see Emken, Kouvaris & Shoemaker [1702.07750]

Takes deflection into account in a thin portion of Earth's crust:





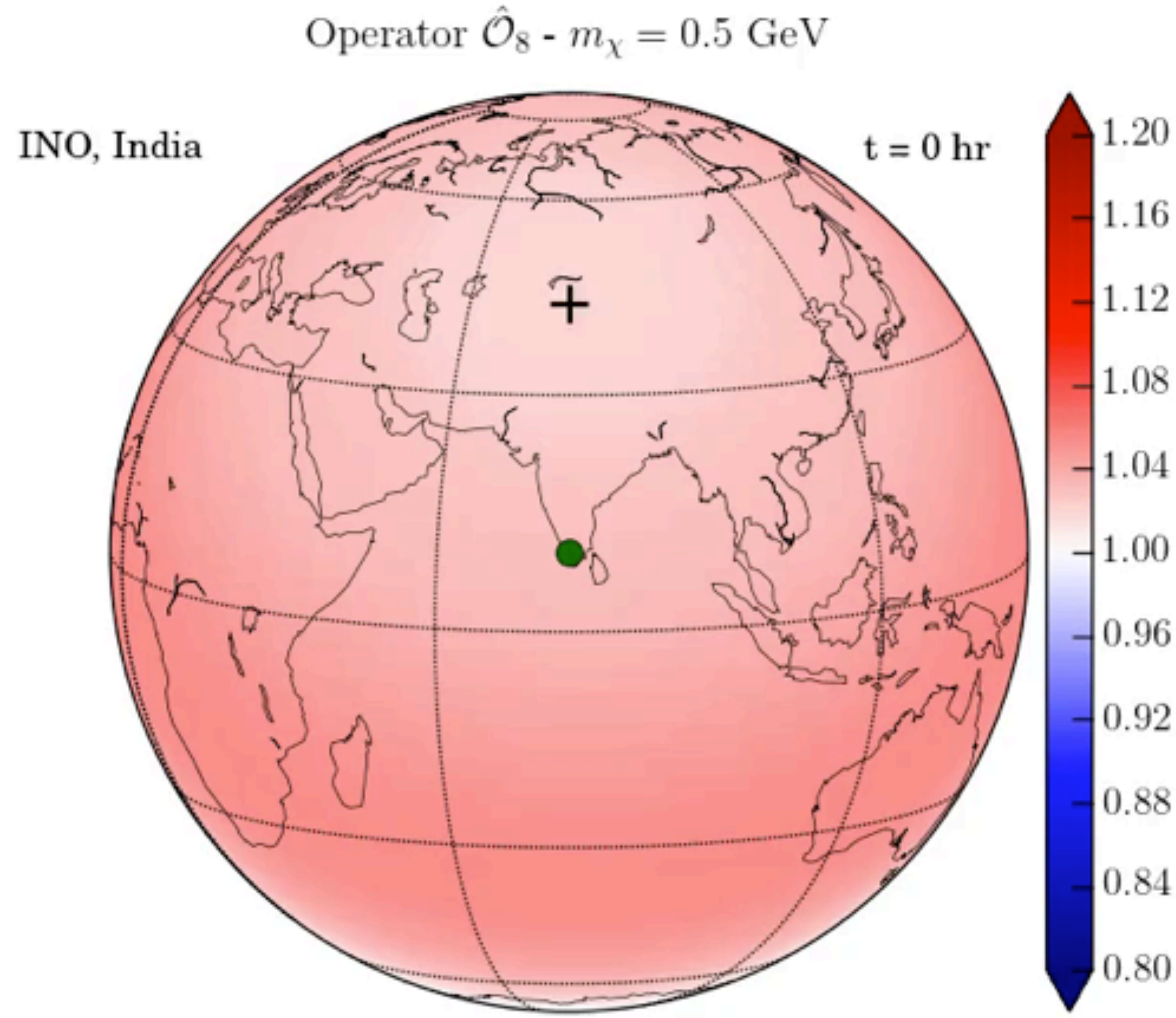
But still need analytic calculations to test and calibrate!

Future ideas

- **Mapping out the parameter space** - What are the signatures of different DM-nucleon (DM-e? Long-range?) interactions?
- **Directional signatures of Earth-Scattering** - Does directional sensitivity enhance these effects?
- **The impact of astrophysical uncertainties** - Would diurnal modulation be a 'clean' signature? What about ρ_χ?
- **Low mass DM** - Can we make progress in the diffusion regime?
- **High mass DM** - WIMPzilla trajectories should be simpler. What are the signatures for heavy SIMP DM?
- **Monte Carlo simulations** - Can we tackle Earth-Scattering for an arbitrary point in parameter space? How can we test/calibrate these simulations?

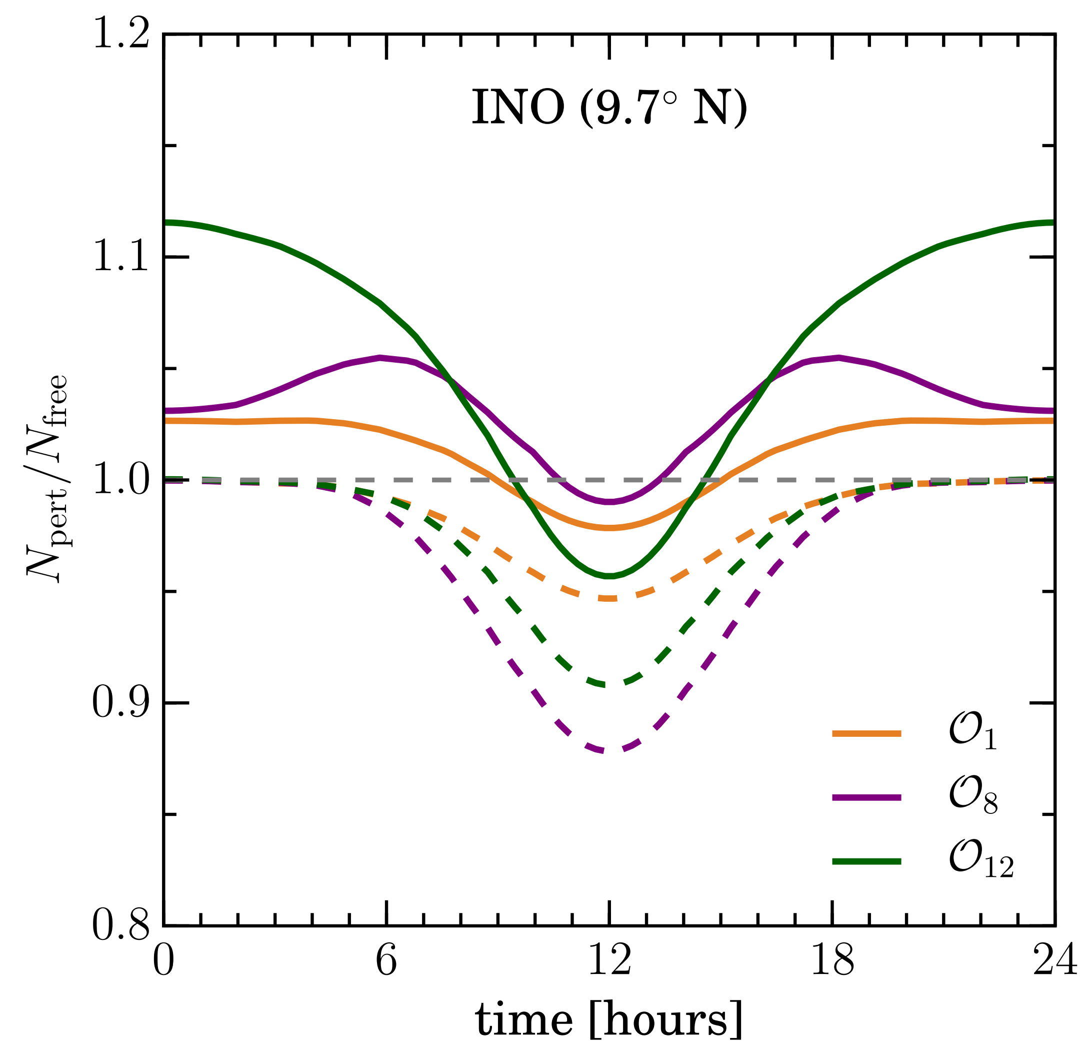
Backup Slides

INO - Operator 8

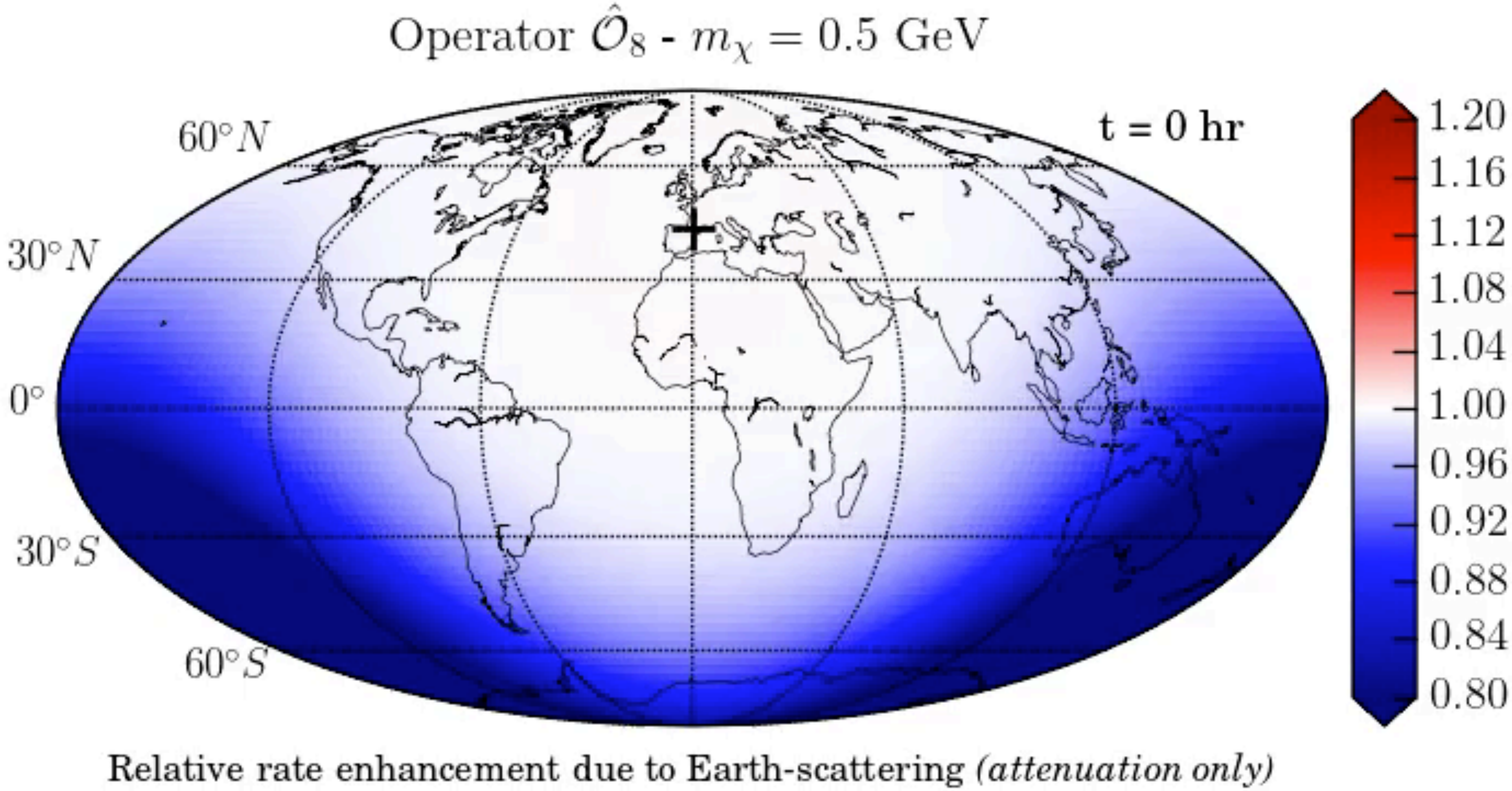


Relative rate enhancement due to Earth-scattering *(attenuation + deflection)*

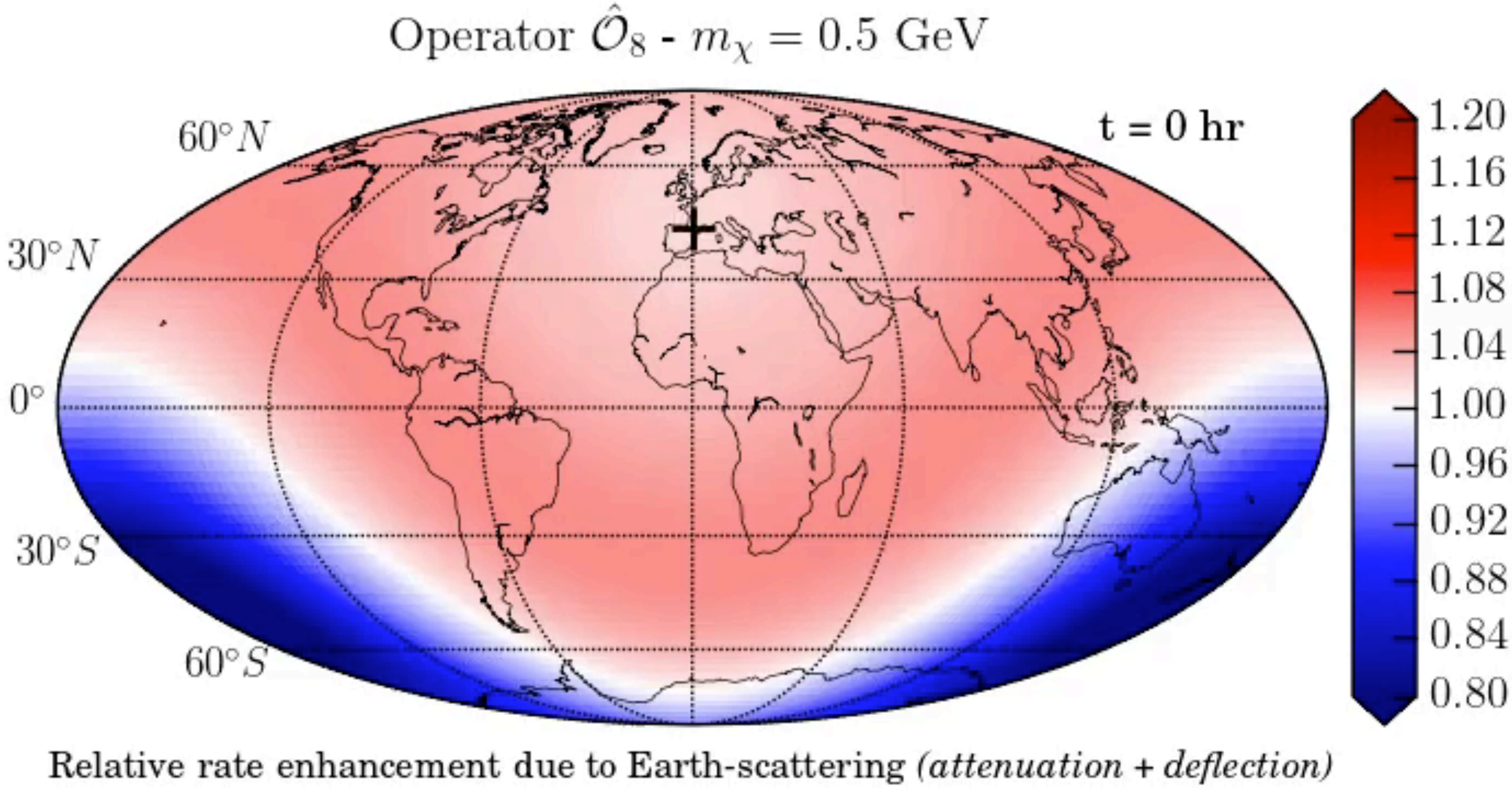
Operator O8



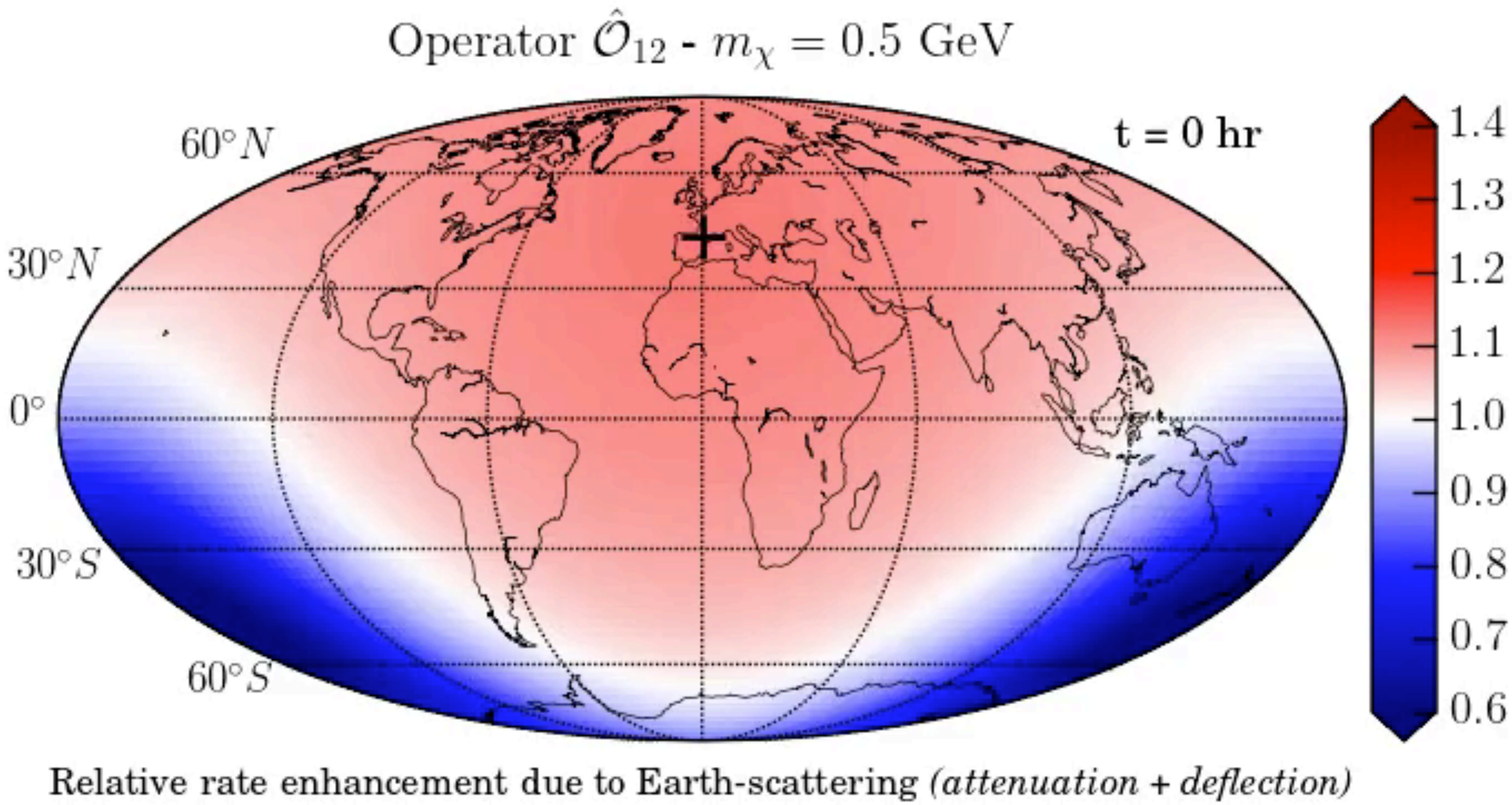
Mapping the CRESST-II Rate



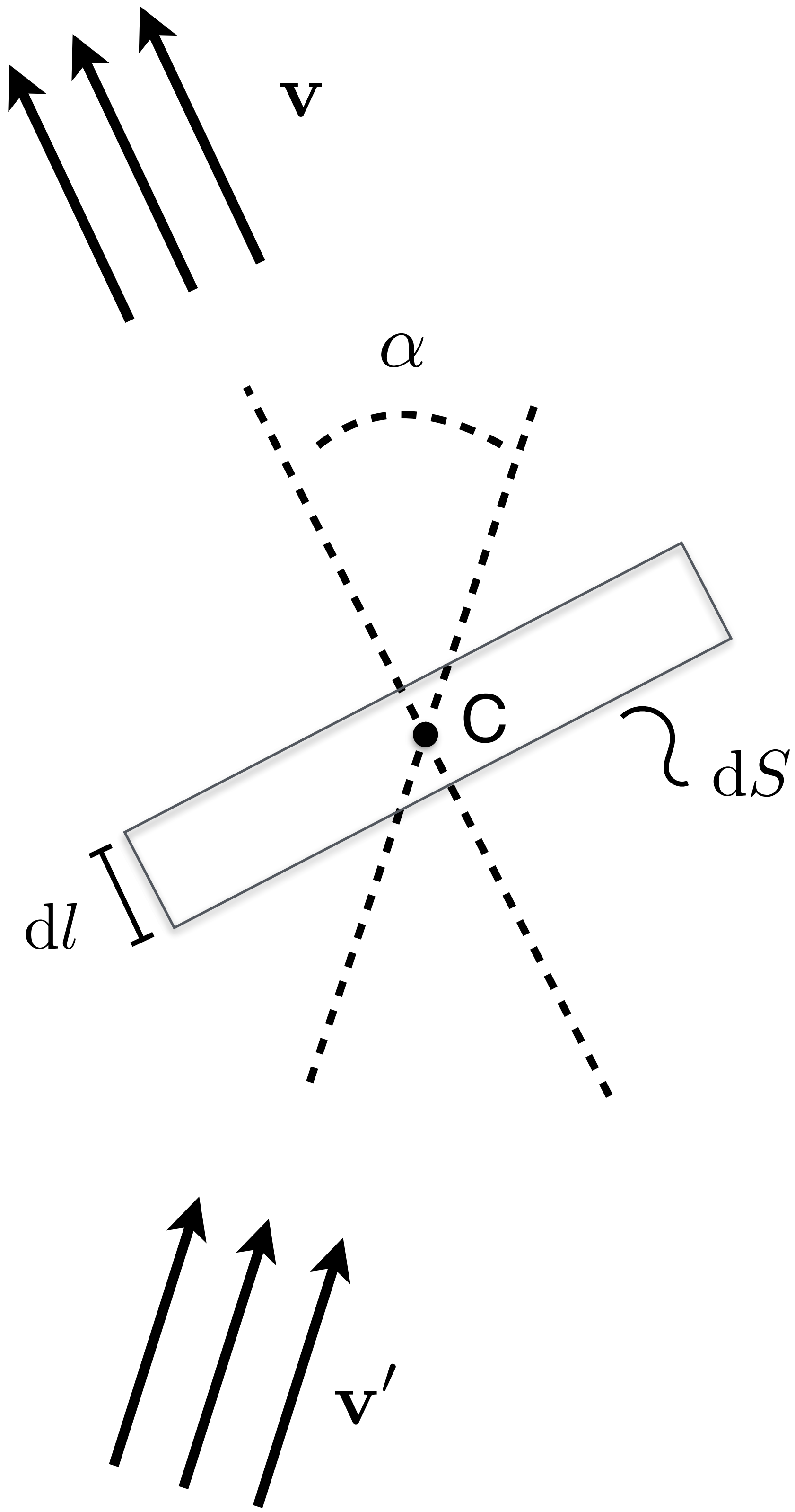
Mapping the CRESST-II Rate



Mapping the CRESST-II Rate



Deflection



Rate of particles entering the region:

$$n_\chi\, f_0(\mathbf{v}')\, v' \cos\alpha\, \mathrm{d}S\, \mathrm{d}^3\mathbf{v}'$$

Probability of scattering in the region:

$$\frac{\mathrm{d}l}{\lambda_i(\mathbf{r}, v')\cos\alpha}\, P(\mathbf{v}' \to \mathbf{v})\, \mathrm{d}^3\mathbf{v}$$

Rate of particles leaving the region:

$$n_\chi f_D(\mathbf{v})\, v\, \mathrm{d}S\, \mathrm{d}^3\mathbf{v}$$

$$\downarrow$$

Deflected velocity distribution:

$$f_D(\mathbf{v}) = \frac{\mathrm{d}l}{\lambda_i(\mathbf{r}, v')}\frac{v'}{v} f_0(\mathbf{v}') P(\mathbf{v}' \to \mathbf{v})\, \mathrm{d}^3\mathbf{v}'$$

Deflection

Deflected velocity distribution (from a single point):

$$f_D(\mathbf{v}) = \frac{\mathrm{d}l}{\lambda_i(\mathbf{r}, v')} \frac{v'}{v} f_0(\mathbf{v}') P(\mathbf{v}' \rightarrow \mathbf{v}) \, \mathrm{d}^3\mathbf{v}'$$

Probability of scattering from one velocity to another can be written:

$$\begin{aligned} P(\mathbf{v}' \rightarrow \mathbf{v}) &= \frac{1}{2\pi} \frac{1}{v^2} \delta(v - v'/\kappa_i) \, P(\cos\alpha) \\ &= \frac{1}{2\pi} \frac{v'}{v^3} \delta(v' - \kappa_i v) \, P(\cos\alpha) \end{aligned}$$

$v'/v \equiv \kappa_i$

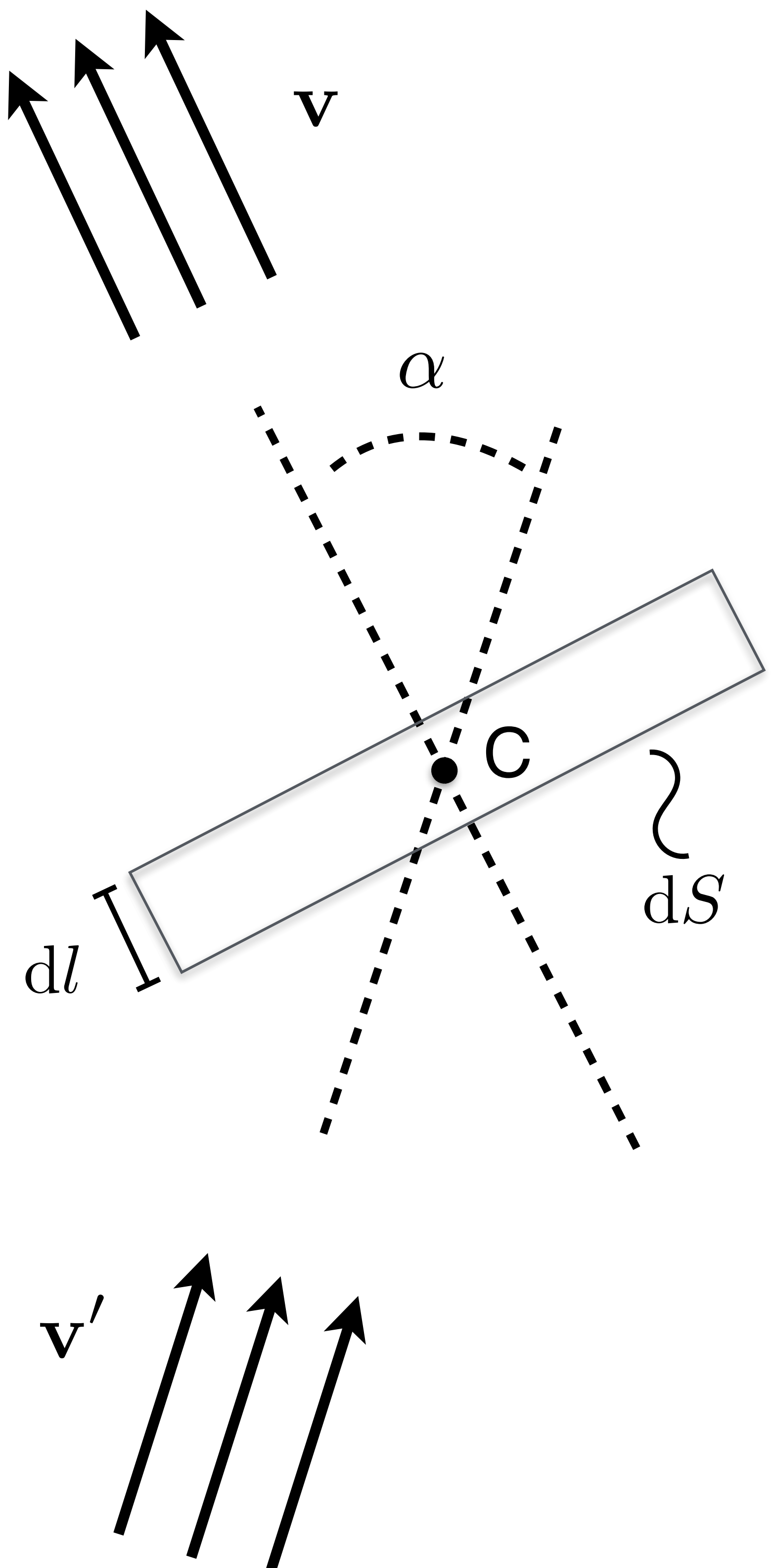
fixed by kinematics (for a given α)

Need to integrate over all incoming velocities and over all points C:

$$f_D(\mathbf{v}) = \frac{1}{2\pi} \int_{AB} \frac{\mathrm{d}l}{\lambda_i(\mathbf{r}, v')} \int \mathrm{d}^3\mathbf{v}' \, \frac{v'^2}{v^4} \delta(v' - \kappa_i v) f_0(v', \hat{\mathbf{v}}') P_i(\cos\alpha)$$

Collect everything together, and sum over Earth species…

Deflection



Equate rate of particles entering and leaving region, having scattered…

Then integrate over all incoming velocities and over all points C:

$$f_D(\mathbf{v}) = \frac{1}{2\pi}\int_{AB} \frac{\mathrm{d}l}{\lambda_i(\mathbf{r}, v')} \int \mathrm{d}^3\mathbf{v}' \, \frac{v'^2}{v^4} f_0(v', \hat{\mathbf{v}}') P_i(\cos\alpha)$$

$$v'/v \equiv \kappa_i$$

fixed by kinematics
(for a given α)

Collect everything together, and sum over Earth species…

[Detailed calculation in the paper]